Direct multi-parameter inversion of geo-radar data through inverse scattering

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ABSTRACT

Multidimensional variations in both dielectric permittivity and conductivity may in principle be directly determined from geo-radar reflection data using an inverse scattering formulation. A Born series expansion of electric field data (using Green’s functions appropriate for a homogeneous reference medium) is inverted, order by order, for a scattering potential comprised of perturbations in permittivity and conductivity. The linear component of the inversion is a crucial step, regardless of which of its two possible uses it is put to: if the perturbations are small and transient, as an estimate of the actual medium variations (i.e., the inverse Born approximation) and a stopping point; or, if the perturbations are large and extended, as essentially altered data to be used as input to the higher order inverse terms. Its form depends on the nature and dimensionality of the reference and actual media: distinct linear algorithms are needed for all combinations of 1D v. 2D perturbations and dielectric/dielectric, dielectric/conductive, and conductive/conductive pairs of reference/actual media. The directness of the approach clarifies that single source/multiple receiver (or single receiver/multiple source) data are required to determine arbitrary conductivity and permittivity profiles, and multiple sources and receivers are required to determine arbitrary 2D variations of these two parameters. Numerical tests, on synthetic data generated using a published finite difference forward modeling code, illustrate the recovery to first order of locations and amplitudes of a set of simple dielectric permittivity and conductivity models. The results of this preliminary study are encouraging and suggest extension to the full vector geo-radar problem is warranted.

1. Introduction

We develop an inverse scattering framework for treatment of the reflection geo-radar problem, connecting (or beginning to connect) to this near-surface methodology a set of distinct multi-parameter and multi-dimensional inverse tools originally developed for seismic exploration: e.g., direct linear inverse scattering in the style of Clayton and Stolt (1981), migration-inversion as described by Stolt and Weglein (1985); Weglein and Stolt (1999), and direct non-linear inverse scattering (Weglein et al., 1981; Stolt and Jacobs, 1980; Weglein et al., 2003).

Geo-radar and seismology are already closely linked, with related wave equations and similar phases detected in their respective data (Bohidar and Hermance, 2002; Nobes et al., 2005). Many geo-radar imaging methods (Fisher et al., 1992; Nemeth et al., 1996; van der Kruk et al., 2003; Grasmueck et al., 2005; Sena et al., 2006; Streich et al., 2007; Irving et al., 2007), inversion methods (Saintenoy and Tarantola, 2001; Day-Lewis et al., 2006; Bradford, 2006; Clement and Knoll, 2006; Di et al., 2006; Ernst et al., 2007; Buursink et al., 2008), and other processing methods such as dispersion removal (Irving and Knight, 2003) have been derived similarly to existing reflection or cross-well seismic techniques. Scattering theory has also been applied to the geo-radar problem. Linear scattering formulations have been used to derive geo-radar wave field extrapolators (Kruk et al., 2003) and Generalized Radon Transform methods (Wang and Ongistaglio, 2000), and are mentioned in the context of asymptotic inversion by (Bleistein et al., 2000); also, Routh and Johnson (2005) have presented a fully non-linear inverse scattering method for geo-radar data. The contribution we make here (originally presented at the SEG Annual Meeting in Las Vegas, NV by Innanen and Routh, 2008) lies in (1) what is shared with migration-inversion, i.e., that the inverse output is expressed as an exact set of transforms and weights applied to the input data, wherein the wave field is brought down to depth, and the local angle/wavenumber dependence of the field at depth is used to estimate multiple parameters, (2) what is shared with non-linear inverse scattering series methods. In adding a direct inverse method to the set of inverse methods referenced above, the authors feel that as well as presenting a set of algorithms, we are building a framework from within which insight into the general character of the inverse GPR reflection problem may be derived.

The linear part of the full nonlinear inverse scattering problem is interpretable in two ways, as either an end result, providing an estimate of changes in permittivity and conductivity, or as an input to higher order components of the inverse scattering problem. Which of the two is appropriate depends in large part on the size and extent of the perturbation. In this paper we study the linear problem in detail, and remain mostly unbiased in this regard, but amongst the numerical examples we do include some large contrast cases that illustrate the “work left to do” by the non-linear parts of the inverse theory. Detailed theory and examples of that work is left for a future communication. We further assume a scalar wave equation adequately describes the radar field in the subsurface, and we consider the data to be measurements of a single component of the electric field. This is of course an oversimplification, but one which allows us to initially consider the details of the inverse scattering geo-radar problem in a tractable environment. Extension to the full vector geo-radar problem is a natural next step, in
particular if field tests indicate that the scalar approximation is inadequate. Seismic aspects of linearized elastic inversion are discussed by Weglein et al. (2009).

This paper is organized as follows. We begin by expressing all relevant scattering quantities, including particular geo-radar wave equations, perturbations, and Green’s functions for homogeneous dielectric and conductive reference media. We then briefly express the general non-linear forward (Born) and inverse scattering series, the latter generated similarly to the reflection seismic derivation of Carvalho (1992), and discussed in detail in the context of seismic exploration by Weglein et al. (1997, 2003). Next we begin a systematic treatment of the linear inverse component of the problem, following the basic approach used by Clayton and Stolt (1981) for the seismic problem. Six distinct cases are considered, corresponding to 1D vs. 2D media (the 3D case not being significantly different in theory from the 2D case), and dielectric/dielectric, dielectric/conductive, and conductive/conductive cases of the reference/perturbed medium. We lastly consider numerically the linear estimation of single- and multiple-parameter profiles and 2D models, and discuss the data requirements for each case.

2. Scattering quantities

We consider two Earth media; a homogeneous reference medium, and a heterogeneous perturbed medium. Within these media a single component of the electric field Green’s function in the radar experiment is assumed to satisfy

\[ \nabla^2 + k^2 E_0(x,x_s,\omega) = \delta(x-x_s), \]  

or, in operator notation, \( \mathbf{L}_0 E_0 = I \) in the reference medium, and

\[ \nabla^2 + K^2 E(x,x_s,\omega) = \delta(x-x_s), \]  

or \( \mathbf{L} E = I \) in the actual medium. The variables \( x, x_s \) and \( \omega \) are the spatial locations of the receiver, source, and the temporal angular frequency respectively (Figure 1). The most general medium we consider has propagation constants defined by

\[ K^2 = -i\omega\mu_0\sigma(x) + \omega^2\mu_0\varepsilon(x), \]

\[ k^2 = -i\omega\mu_0\sigma_0 + \omega^2\mu_0\varepsilon_0, \]  

where \( \varepsilon, \varepsilon_0 \) are the actual and reference frequency-independent dielectric permittivities respectively, \( \sigma, \sigma_0 \) are the actual and reference frequency-independent conductivities respectively, and \( \mu_0 \) is the reference magnetic permeability, which we assume to be constant and known. The scattering potential \( V \) has the form

\[ \mathbf{V} = \mathbf{L}_0 - \mathbf{L} = k^2 - K^2 = -i\omega\mu_0[\sigma_0 - \sigma(x)] + \omega^2\mu_0[\varepsilon_0 - \varepsilon(x)], \]  

i.e., it is straightforwardly and linearly related to the medium parameter perturbations. In 2D (i.e., for line sources and receivers), the reference Green’s functions \( E_0 \) may be written (De Santo, 1992)

\[ E_0(x_s,z_s|x_s,z_s,\omega) = \frac{1}{2\pi} \int dk s ' e^{ik s 'x_g} E^0 \left| \frac{e^{iq s 'z_g}}{i2q s \theta} \right|, \]

for a source at \( (x_s,z_s) \) and a receiver at \( (x_g,z_g) \) (see Figure 1), where \( q' = (k^2 - k_{s g}^2)^{1/2} \). Alternatively, having undergone a Fourier transform over \( x_g \), \( E_0 \) may be expressed as

\[ E_0(x_g,z_g|x_s,z_s,\omega) = e^{ik_{s g}x_g} E^0 \left| \frac{e^{iq s z_g}}{i2q s \theta} \right|, \]

or over \( x_s \), as

\[ E_0(k_{s g}x_s,z_s,x_g,\omega) = e^{-ik_{s g}x_g} E^0 \left| \frac{e^{iq s z_g}}{i2q s \theta} \right|, \]

where \( k_{s g} \) and \( k_g \) are the Fourier conjugates of \( x_s \) and \( x_g \) respectively, and the quantities \( q \) are constructions resembling depth wavenumbers, e.g., \( q_s = (k^2 - k_{s g}^2)^{1/2} \) and \( k^2 \) is given by equation (3); we illustrate the underlying plane wave geometry in Figure 2. We use the Fourier transform
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and

\[ q_g = \sqrt{\omega^2 \mu_0 \varepsilon_0 - k_g^2} \]  \hspace{1cm} (13)

2.1.1 1D vs. 2D media

If the actual medium is also dielectric we have in effect a single parameter problem. The propagation constants become

\[ K^2 = \omega^2 \mu_0 \varepsilon, \]
\[ k^2 = \omega^2 \mu_0 \varepsilon_0 \]  \hspace{1cm} (14)

hence \( V \) takes on the form

\[ V = \omega^2 \mu_0 \varepsilon_0 \xi \]  \hspace{1cm} (15)

where we define

\[ \xi(z) = 1 - \frac{\varepsilon(z)}{\varepsilon_0} \]  \hspace{1cm} (16)

in 1D, and

\[ \xi(x,z) = 1 - \frac{\varepsilon(x,z)}{\varepsilon_0} \]  \hspace{1cm} (17)

in 2D.

2.2 Dielectric reference medium, conductive actual medium

For a dielectric reference medium, the Green’s function quantities remain of the form in equations (5)-(13).

2.2.1 1D vs. 2D media

The situation is nevertheless distinct because of the form of the actual medium:

\[ K^2 = -i \omega \mu_0 \sigma + \omega^2 \mu_0 \varepsilon \]  \hspace{1cm} (18)

as compared with the reference medium:

\[ k^2 = \omega^2 \mu_0 \varepsilon_0 \]  \hspace{1cm} (19)

from which we define

\[ V = \omega^2 \mu_0 \varepsilon_0 \xi + i \omega \mu_0 \gamma \]  \hspace{1cm} (20)

where in 1D

\[ \xi(z) = 1 - \frac{\varepsilon(z)}{\varepsilon_0} \]
\[ \gamma(z) = \sigma(z) \]  \hspace{1cm} (21)

and in 2D

\[ \xi(x,z) = 1 - \frac{\varepsilon(x,z)}{\varepsilon_0} \]
\[ \gamma(x,z) = \sigma(x,z) \]  \hspace{1cm} (22)

For convenience \( \gamma \) has been chosen not to be a dimensionless quantity, which all other perturbations in this paper are; here it has units of conductivity.
2.3 Conductive reference medium, conductive actual medium

If the reference medium is conductive, the \( q \) quantities in the reference Green’s functions are altered, notably becoming complex:
\[
q' = \sqrt{-i\omega\mu_0\sigma_0 + \omega^2\mu_0\varepsilon_0 - k'^2} \\
q = \sqrt{-i\omega\mu_0\sigma_0 + \omega^2\mu_0\varepsilon_0 - k^2} \\
q_s = \sqrt{-i\omega\mu_0\sigma_0 + \omega^2\mu_0\varepsilon_0 - k_s^2}
\]

and
\[
q_s = \sqrt{-i\omega\mu_0\sigma_0 + \omega^2\mu_0\varepsilon_0 - k_s^2}.
\]

2.3.1 1D vs. 2D media

The form of the perturbations differ also, since we have for propagation constants:
\[
K^2 = -i\omega\mu_0\sigma + \omega^2\mu_0\varepsilon k^2 \\
k^2 = -i\omega\mu_0\sigma_0 + \omega^2\mu_0\varepsilon_0
\]
and consequently
\[
V = \omega^2\mu_0\varepsilon_0\xi + i\omega\mu_0\sigma_0\gamma
\]
where in 1D
\[
\xi(z) = 1 - \frac{\varepsilon(z)}{\varepsilon_0}, \\
\gamma(z) = 1 - \frac{\sigma(z)}{\sigma_0}
\]
and in 2D
\[
\xi(x,z) = 1 - \frac{\varepsilon(x,z)}{\varepsilon_0}, \\
\gamma(x,z) = 1 - \frac{\sigma(x,z)}{\sigma_0}.
\]

In this case the perturbation \( \gamma \) has been defined as a dimensionless quantity similarly to \( \xi \).

3. Forward scattering for geo-radar

The Born series (Taylor, 1972; Joachain, 1975; Goldberger and Watson, 2004) arises from the Lippmann-Schwinger equation, an identity relating equations (1), (2) and (4). In operator notation the Lippmann-Schwinger equation has the form
\[
E = E_0 + E_0VE,
\]
and in integral notation, the form
\[
E(x,x',\omega) = E_0(x,x',\omega) + \int d^3x' E_0(x,x',\omega)V(x')E(x',x',\omega) ,
\]
which after being repeatedly substituted back onto itself, produces the Born series
\[
E(x,x',\omega) = E_0(x,x',\omega) + E_1(x,x',\omega) + E_2(x,x',\omega) + \ldots
\]
where
\[
E_1(x,x',\omega) = \int d^3x' E_0(x,x',\omega)V(x')E_0(x',x',\omega) ,
\]

\[
E_2(x,x',\omega) = \int d^3x' E_0(x,x',\omega)V(x')E_0(x',x',\omega) ,
\]

et al., 2003), we begin by taking the Born series and restricting the scattered field to lie on the measurement surface, thus defining the measured data \( D' \):
\[
D'(x,x',\omega) = \left[ E(x,x',\omega) - E_0(x,x',\omega) \right]_{ms} \\
= \left[ E_1(x,x',\omega) + E_2(x,x',\omega) + \ldots \right]_{ms} .
\]

We make the assumption that the source wavelet to be known and deconvolved, leaving intact all important components of the temporal spectrum of the data. This is of course a large assumption, but a meaningful treatment of the wavelet problem is beyond the scope of this paper. Taking the ‘ms’ stipulation as necessarily coincident) planes lying above the depth support of \( V \), we next expand the perturbation as a series in orders of the data:
\[
V = V_1 + V_2 + V_3 + \ldots
\]
which, upon substitution into equation (35) and equation of like orders, produces
\[
D'(x,x',\omega) = \int d^3x' E_0(x,x',\omega)V_1(x')E_0(x',x',\omega) \quad (37)
\]
\[
= -\frac{1}{\omega^2\mu_0\varepsilon_0}\int d^3x' E_0(x,x',\omega)V_1(x') \quad (38)
\]
etc. This formalism permits that \( V_1 \) is derivable directly from the data \( D' \) and reference medium information, and all \( V_n \) are derivable directly from reference medium information and \( V_1 \) (Weglein et al., 2003). The inverse solution is produced by summing the \( V_n \) as in equation (36); the linearization \( V_1 = V \) is the inverse Born approximation.

5. Linear inverse scattering for geo-radar

We consider the linear component of the non-linear inverse scattering problem and its application to reflected geo-radar data, generally following the approach applied by Clayton and Stolt (1981) to the seismic inverse problem. The change of variables to pseudo-depth and plane wave angle has been of particular value for direct non-linear inverse scattering series seismic imaging and target identification methods (Weglein et al., 2001; Shaw et al., 2004; Shaw, 2005; Zhang and Weglein, 2005; Liu et al., 2006; Zhang, 2006; Liu, 2006; Weglein, 2008). Because of the absorption and dispersion, there is also some resemblance to the seismic inverse developments of Innanen and Weglein (2007).
5.1 1D media

We next pose the 1D 2-parameter linear inverse problem for computing $D$ from $D'$. We hide some details by defining a new $D'$ to be $D'(x_g',z_g',x_s',z_s') = e^{-i\theta}D(x_g,z_g,x_s,z_s)$, where $k_z = \frac{q_s}{-2\theta}$. Equation (45) realized over $N$ values of plane wave incident angle $\theta$, may be written:

$$d = F_2 \begin{bmatrix} \xi_1(k_z|\Theta_1) \\ \gamma_1(k_z|\Theta_1) \end{bmatrix}$$


5.1.1 Dielectric reference medium, dielectric actual medium

The reflection data $D'$ is measured at one receiver location $(x_g, z_g)$ and many shot locations $(x_s, z_s)$; it is then Fourier transformed over time and lateral shot location. Making use of the quantities defined in equations (5)–(7), (11)–(13), and (16), the linear relationship in equation (37) becomes

$$D'(x_g',z_g',x_s',z_s') = -\frac{q_s^2}{2\theta \mu_0 \varepsilon_0} D(x_g',z_g',x_s',z_s')e^{i\theta}$$

so that, after a change of variables equation (39) takes on the simple form

$$D(k_z,\theta) = \xi_1(k_z|\theta)$$

where

$$k_z = -2\theta$$

$$\theta = \cos^{-1}\frac{q_s}{k}$$

This permits us to construct a profile $\xi_1(z)$ for any plane wave angle of incidence $\theta$, available in the data, by calculating the inverse Fourier transform:

$$\xi_1(z|\theta) = \frac{1}{2\pi} \int dk_z e^{ik_zz} \xi_1(k_z|\theta)$$

5.1.2 Dielectric reference medium, conductive actual medium

We next pose the 1D 2-parameter linear inverse problem for a dielectric reference medium, using the quantities defined in equations (5)–(7), (11)–(13), and (21). We have

$$D'(x_g',z_g',x_s',z_s') = -\frac{q_s^2}{2\theta \mu_0 \varepsilon_0} \left[ \omega^2 \mu_0 \varepsilon_0 \xi_1(-2\theta) + i\omega \mu_0 \gamma_1(-2\theta) \right]$$

Again computing $D$ from $D'$ using equation (40), we change variables using equation (42), obtaining

$$D(k_z,\theta) = \xi_1(k_z)+F_3(k_z,\theta)\gamma(k_z)$$

where

$$F_3(k_z,\theta) = \left[ i\omega(k_z,\theta) \varepsilon_0 \right]^{-1}$$

$$\omega(k_z,\theta) = \frac{k_z \sqrt{\mu_0 \varepsilon_0}}{2\cos \theta}$$

Equation (54) realized over a range of $N$ frequencies becomes

$$d = F_3 \begin{bmatrix} \xi_1(k_z|\Theta) \\ \gamma_1(k_z|\Theta) \end{bmatrix}$$

with

$$F_3 = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ F_2(k_z,\theta_1) & F_2(k_z,\theta_2) & \ldots & F_2(k_z,\theta_{SN}) \end{bmatrix}^T$$

Finally, for any given $\Theta$, we may determine profiles

$$\xi_1(z|\Theta) = \frac{1}{2\pi} \int dk_z e^{ik_zz} \xi_1(k_z|\Theta)$$

$$\gamma_1(z|\Theta) = \frac{1}{2\pi} \int dk_z e^{ik_zz} \gamma_1(k_z|\Theta)$$

through inverse Fourier transforms.

5.1.3 Conductive reference medium, conductive actual medium

We next pose the 1D 2-parameter linear inverse problem for a conductive reference medium, using the quantities defined in equations (5)–(7), (23)–(25), and (28). We have

$$D'(x_g',z_g',x_s',z_s') = -\frac{q_s^2}{2\theta \mu_0 \varepsilon_0} \left[ \omega^2 \mu_0 \varepsilon_0 \xi_1(-2\theta) + i\omega \mu_0 \gamma_1(-2\theta) \right]$$

which, after calculating $D$ from $D'$ via equation (40), and employing a slightly different change of variables, from $(k_z,\omega)$ to $(k_s,\omega)$, leads to the form

$$D(k_z,\omega) = \xi_1(k_z)+F_3(\omega)\gamma(k_z)$$

where $k_s$ is defined in equation (40), and

$$F_3(\omega) = \sigma_0(i\omega \varepsilon_0)^{-1}$$

Practically speaking the plane wave incidence angle $\theta$, could still have been used as before, but it is less convenient than $\alpha$ for the absorptive reference case. Equation (54) realized over a range of $N$ frequencies becomes

$$d = F_3 \begin{bmatrix} \xi_1(k_z|\Theta) \\ \gamma_1(k_z|\Theta) \end{bmatrix}$$

where

$$F_3 = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ F_2(k_s,\alpha_1) & F_2(k_s,\alpha_2) & \ldots & F_2(k_s,\alpha_{SN}) \end{bmatrix}^T$$
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\[ F_3 = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ F_3(\omega_1, \ldots, \omega_N) \\ \end{bmatrix}^T \]  \hspace{1cm} (57)

and

\[ d = \begin{bmatrix} D(k_z, \omega_1), D(k_z, \omega_2), \ldots, D(k_z, \omega_N) \end{bmatrix}^T \]  \hspace{1cm} (58)

and \( \Omega = \{ \omega_1, \omega_2, \ldots, \omega_N \} \). Provided \( N > 2 \), we may determine the unknowns via, e.g.,

\[ \begin{bmatrix} \xi_1(k_z|\Omega) \\ \gamma_1(k_z|\Omega) \end{bmatrix} = (F_3^H F_3)^{-1} F_3^H d . \]  \hspace{1cm} (59)

The wavenumber \( k_z \) in this case has an imaginary component, since for the conductive reference medium \( \Omega_0 \neq 0 \). For this reason, the final step, which in constructing the profiles involves an integration over \( \exp(ik_zz) \), will tend to be unstable and sensitive to noise. Without being specific in this paper, we advocate the introduction of a tapering function \( W(k_z) \), to be designed with an appropriate inverse filtering argument to stabilize the integral, while optimally constructing profiles that honor the data.

Then we have:

\[ \xi_1(z|\Omega, W) = \frac{1}{2\pi} \int dk_z e^{ik_zz} W(k_z) \xi_1(k_z|\Omega_s) \]

\[ \gamma_1(z|\Omega, W) = \frac{1}{2\pi} \int dk_z e^{ik_zz} W(k_z) \gamma_1(k_z|\Omega_s) . \]  \hspace{1cm} (60)

### 5.2 2D media

We next consider the linear inverse scattering problem for line radar sources and receivers over vertically and horizontally varying media. We assume the presence of multiple shot and receiver records, i.e., a dense surface population of sources and receivers. We continue to closely follow Clayton and Stolt (1981), using the Fourier variable definitions:

\[ k_m = k_g - k_s \]

\[ k_z = -q_g - q_s \]

\[ k_h = q_g + q_s \]  \hspace{1cm} (61)

and the relations (Clayton and Stolt’s equations 31–33):

\[ q_g(k_m, k_z, k_h) = -\frac{k_z}{2} \left( 1 - \frac{k_m k_h}{k_z^2} \right) \]

\[ q_s(k_m, k_z, k_h) = -\frac{k_z}{2} \left( 1 + \frac{k_m k_h}{k_z^2} \right) \]  \hspace{1cm} (62)

\[ \omega(k_m, k_z, k_h) = -\frac{k_z}{2\sqrt{\mu_0 e_0}} \sqrt{\left( 1 + \frac{k_m^2}{k_z^2} \right) \left( 1 + \frac{k_h^2}{k_z^2} \right)} \]

which have been modified slightly for the radar problem.

#### 5.2.1 Dielectric reference medium, dielectric actual medium

We begin by posing the 2D 1-parameter linear inverse problem for a dielectric reference and actual medium, using the quantities defined in equations (5)–(7), (11)–(13), and (17). We have

\[ D'(x_g, z_g, k_m, z_s) = -\frac{e^{iq_g z_g - iq_s z_s}}{4q_g q_s} \omega^2 \mu_0 e_0 \xi_1(k_g - k_m, q_g - q_s) \]  \hspace{1cm} (63)

\[ D(k_m, z_g, k_s, z_s) = -4 \frac{q_g q_s}{\omega^2 \mu_0 e_0} \omega^2 \mu_0 e_0 \xi_1(k_g - k_m, q_g - q_s) \]  \hspace{1cm} (64)

and changing variables according to equations (61)–(62), we have

\[ D(k_m, k_z, k_h) = \xi_1(k_m, k_z, k_h) , \]  \hspace{1cm} (65)

after which the linear dielectric permittivity perturbation model may be constructed through the inverse Fourier transform

\[ \xi_1(x, z|k_h) = \frac{1}{2\pi} \int dk_m \int dk_z e^{ik_m x + ik_z z} \xi_1(k_m, k_z|k_h) \]  \hspace{1cm} (66)

for any available \( k_h \) in the data.

#### 5.2.2 Dielectric reference medium, conductive actual medium

We next pose the 2D 2-parameter linear inverse problem for a dielectric reference medium and a conductive actual medium, using the quantities defined in equations (5)–(7), (11)–(13), and (22). We have

\[ D'(x_g, z_g, k_m, k_z) = -\frac{e^{iq_g z_g - iq_s z_s}}{4q_g q_s} \omega^2 \mu_0 e_0 \xi_1(k_g - k_m, k_z) \]  \hspace{1cm} (63)

\[ D(k_m, k_z, k_h) = -\frac{e^{iq_g z_g - iq_s z_s}}{4q_g q_s} \omega^2 \mu_0 e_0 \xi_1(k_g - k_m, k_z) \]  \hspace{1cm} (64)

and the totality of utilized \( k_h \) values is represented by

\[ \{ k_{h1}, k_{h2}, \ldots, k_{hN} \} \]  \hspace{1cm} (73)
Again one option is to solve this system with a least-squares approach:

\[
\begin{bmatrix}
\xi_1(k_m,k_z|k_h) \\
\gamma_1(k_m,k_z|k_h)
\end{bmatrix}_{\text{lsq}} = (F^H_4F_4)^{-1}F^H_4d ,
\]

after which the linear perturbations may be constructed via

\[
\begin{align*}
\xi_1(x,z|k_h) &= \frac{1}{2\pi} \int dk_m \int dk_x e^{ik_x x} e^{ik_z z} \xi_1(k_m,k_z|k_h) \\
\gamma_1(x,z|k_h) &= \frac{1}{2\pi} \int dk_m \int dk_x e^{ik_x x} e^{ik_z z} \gamma_1(k_m,k_z|k_h)
\end{align*}
\]

So, again making use of equation (64), and under the same 2D change of variables, we obtain

\[
D(k_m,k_z,k_h) = \xi_1(k_m,k_z|k_h) + F_5(k_m,k_z,k_h)\gamma_1(k_m,k_z|k_h),
\]

where

\[
F_5(k_m,k_z,k_h) = -\sigma_0[I \omega(k_m,k_z,k_h)\varepsilon_0]^{-1} .
\]

As before, for a set of 2$N$ data points with fixed $k_m$ and varying $k_h$, we may write

\[
d = F_5\begin{bmatrix}
\xi_1(k_m,k_z|k_h) \\
\gamma_1(k_m,k_z|k_h)
\end{bmatrix}_{\text{lsq}},
\]

5.2.3 Conductive reference medium, conductive actual medium

We next pose the 2D 2-parameter linear inverse problem for a conductive reference medium, using the quantities defined in equations (5)–(7), (23)–(26), and (29). We have

\[
D'(x,z,k_m,k_z,\omega) = -\frac{\varepsilon_0\mu_0\sigma_0}{4\varepsilon_0\mu_0} \left[ \omega^2\mu_0\varepsilon_0 \xi_1(k_m,k_z,\omega) - q_s - q_s \right]
\]

and

\[
\xi_1(x,z|k_m,k_z|k_h) = \frac{1}{2\pi} \int dk_m e^{ik_m x} \xi_1(k_m,k_z|k_h) \\
\gamma_1(x,z|k_m,k_z|k_h) = \frac{1}{2\pi} \int dk_m e^{ik_m x} \gamma_1(k_m,k_z|k_h)
\]

5.3 Numerical examples

In this section we carry out a selection of examples of the above linear inverse procedures, using as input synthetic geo-radar data produced by a published 2D code.

5.3.1 Synthetic data and pre-processing

To generate the data, we use both analytic forms, and, more extensively, the published finite di!erence code of Irving and Knight (2006). The software was run in transverse magnetic mode, to model the y-component of the electric

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\]
field due to line sources and receivers within distributions, in x and z, of dielectric permittivity and conductivity. Figure 4 illustrates the five distinct parameter and source/receiver configurations we use. The first, a single plane interface interrogated by a radar wave at normal incidence, is studied using analytic data; the remaining four, discussed in detail below, are input to the finite difference modeling scheme. Figure 5 illustrates portions of the resulting data for each of the six configurations.

The finite difference data is pre-processed identically in each case. Figure 6 illustrates this two-stage procedure. The top left panel is an example input data record (equivalently, multiple sources and a single receiver or multiple receivers and a single source). First, since the data are defined to be the difference between total and reference fields (see equation (35)), the direct wave is modeled (top right) and removed (middle left); in practice this might alternatively be accomplished through muting. And second, a deterministic deconvolution of the source waveform is carried out (middle right). Here we assume full prior knowledge of the wavelet, which, in the finite difference code of Irving and Knight (2006), is a Blackman-Harris pulse; we have chosen a dominant frequency of 100 MHz. The wavelet is plotted in the bottom left panel, and the zero-offset trace of the data record after the deconvolution is plotted in the bottom right panel.

5.3.2 Determination of $\varepsilon_1$, $\sigma_1$ values when both $\varepsilon$ and $\sigma$ vary at a plane boundary

The separate determination of simultaneous variations in conductivity and permittivity is possible, in an inverse scattering setting, because perturbations in these two quantities create distinct variations in the frequency and wavenumber dependence of the data (see equation (45)). By simplifying the 1D, dielectric/conductive configuration described by equation (45) to correspond to a normally incident plane wave on a single interface of known depth, the problem of separation reduces to the determination of scalar values $\varepsilon_1$ and $\gamma_1$ from the reflection coefficient at the layer at two (or more) frequencies. The absolute value of the reflection coefficient for a contrast in relative permittivity from $\varepsilon_0 = 9$ to $\varepsilon_1 = 7$ and in conductivity from nil to $\sigma_1 = 0.005$ mS/m is illustrated in Figure 5 in the top left panel. In Figure 7 we reconstruct the linear part of the $\varepsilon$ and $\sigma$ contrast for a range of frequency pairs (interpolating the surface plot over coincident frequency values) for two cases. In the right column, the contrasts are reconstructed linearly for the input $\varepsilon$, $\sigma$ values described above. The result is stable over frequency pairs, but maintains a consistent error in comparison to the exact relative $\varepsilon$ and $\sigma$ values (7 and $5 \times 10^{-3}$ mS/m) respectively, an expected linear result. In the left column panels we test the ability of the 2-parameter inverse theory to correctly judge that the contrast involved no conductivity variation.

The exact relative $\varepsilon$ and $\sigma$ values below the interface are 7 and 0 mS/m respectively. At normal incidence with analytic data the procedure responds correctly.
5.3.3 Determination of an \( \varepsilon_e(z) \) profile when both \( \varepsilon \) and \( \sigma \) vary at a plane boundary

In practice the separate determination of simultaneously varying \( \varepsilon \) and \( \sigma \) profiles is a more sensitive enterprise, being the aspect of this theory that relies most critically on the amplitudes in the data. We illustrate this using finite difference data created using the model in Figure 4b, and pre-processed as described above. The raw profile results, calculated using equations (44)–(52), and two angles (0 and 0.1 \( \times \pi \) rad) are illustrated in the left column of Figure (8); they oscillate significantly near the interface, however a simple ad-hoc integration filter to smooth out the variations as shown in the right column makes the results more straightforward to interpret. The (filtered) linearly recovered \( \varepsilon(z) \) falls away from the exact value (bold) and towards the analytically-expected linear value, whereas the linearly recovered \( \sigma(z) \) profile is zero as desired in most regions of the profile, but experiences a transitory, anomalous “bump” at the \( \varepsilon \) interface.

5.3.4 Determination of \( \varepsilon_e(z) \) for layered dielectric media

We next focus on the issue of the determination of subsurface structure, by considering the 1-parameter dielectric problem, and using equations (39)–(43). We use synthetic data corresponding to the models in Figure 4c–d, i.e., single layer variations in \( \varepsilon \), one a “fast” layer (c) and one a “slow” layer (d). Both represent relatively large contrasts away from the reference value. The top panel of Figure 9 illustrates the linearly recovered perturbation profiles for the fast layer over a range of plane wave incidence angles. The depth of the lower reflector is underestimated in this large contrast case, with an error aggravated at increasing angle. In this case it is likely that higher order (non-linear) components of the inverse scattering framework would be called for. The lower panel illustrates the amplitude recovery for the profile at normal incidence, which may be compared with the actual amplitudes (relative \( \varepsilon \) values of 9, 3, and 7, from shallowest to deepest). The top panel of Figure 10 similarly illustrates the linearly recovered perturbation profiles for the slow layer, which overestimates the depth of the second interface, with similar aggravated results at large angle. The bottom panel likewise illustrates the profile recovery at normal incidence (against relative \( \varepsilon \) values of 9, 20 and 7 from shallowest to deepest). Again, given the contrast and extent of the perturbation suggests that the inclusion of higher order inverse scattering terms will be a useful step in this case.

5.3.5 Determination of \( \varepsilon_e(x,z) \) models for 2D dielectric media

We finally consider the 2D problem, for dielectric reference and actual media, using equations (63)–(66), and the model illustrated in Figures 4e–f. Here the contrast in the perturbation away from the reference dielectric permittivity of \( \varepsilon = 9 \) is much smaller, allowing us to illustrate perturbation regimes in which the linear inversion could represent a valid stopping point, i.e., in which we would make the inverse Born approximation. Fixing \( k_0=0 \), we
obtain a linear perturbation model illustrated in the top left panel of Figure 11, which is straightforwardly transformed into recovered $\varepsilon(x,z)$ in the top right panel; profiles from the left side of the recovered model are illustrated in the bottom panels. These indicate some low-wavenumber noise in the recovery, but beyond this, because of the relatively low contrast of the 2D structure away from reference (i.e., $\varepsilon = 8$ away from $z = 9$), the interface locations and perturbation amplitudes are qualitatively close to their actual counterparts, and are likely acceptable as final estimates, counter-indicating the use of higher order non-linear inverse terms.

6. Discussion and conclusions

We have developed a framework for direct linear inverse scattering for reflected geo-radar data as a means to determine subsurface electrical properties, and demonstrated that the problem is theoretically tractable, and at least in synthetic settings numerically so also. This represents a preliminary study, in the sense that we have assumed (1) a scalar wave equation and a single component electric field measurement, (2) a known wavelet that may be deconvolved stably, and (3) dense multi-offset sampling of the radar wave field. Determining which if any of these assumptions is approximately valid for field data when these methods are applied is a matter of near term future research.

We view the key benefit of geo-radar inverse scattering to be its “directness” (in other words, its explicit, formulaic calculation of the perturbation estimates in terms of the measured data). The theory renders a clear link between the number of parameters to be inverted for and the number of dimensions in which they vary, and the data required to determine them in these regimes.

Although the purpose of this research is not to generate optimized inversion code, it may be worth noting that the computational expense of these methods is not high. Since they are based on a direct inverse framework, no forward modeling is carried out (except of course that done to generate the synthetic data), and the operations are generally Fourier or Laplace integrals. All the inversions in this paper were carried out on a laptop computer using MATLAB, with the longest taking a few hours to complete.

One- and two-dimensional inverse scattering problems are discussed in detail herein; 3D extension of the 2D forms is mathematically straightforward. We have focused most of the analysis and all of the numerics on the linear component of the full non-linear inverse scattering problem. However, we have deliberately included examples in which the discrepancy between the linear inverse and the actual parameter profiles is (likely) too large to be neglected. In such situations a variety of remedies are possible, most involving attempts to update the reference medium quantities to decrease the size and extent of the perturbations, followed by a further linear inversion. As a matter of ongoing research we are considering a different route, in which these “imperfect” results are input–in essence, altered data–for higher order parts of the inverse scattering series, i.e., equations (38) and beyond. The ultimate interest is casting, for the geo-radar problem, inverse problems along the lines of the seismic inverse scattering series procedures referred to in the introduction.

References


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