High frequency edge diffraction theory

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ABSTRACT

Diffractions make significant contributions to seismic amplitudes. While they are often not included in ray trace models due to their perceived computational difficulty and expense, this paper describes a tractable method for computing diffractions which could benefit seismic modeling in exploration geophysics. We present formulae related to the diffraction of seismic waves by linear edges in elastodynamic media, based on an extension of the high frequency, zero order Asymptotic Ray Theory (ART) formulation. Theoretical development has been kept to a minimum. Rather, schematics indicating relevant details such as shadow boundaries and boundary rays have been included. The identification of the above is required as the argument of the diffraction coefficient is dependent on the angle between the shadow boundary ray and the diffracted ray or equivalently the difference between the diffracted and reflected arrival times. More will be said about this in the text. The reflected geometrical arrival does not exist in the shadow region and its travel time is required to determine the argument of the diffraction coefficient. Using analytic continuation, within the aforementioned limits of minimal theoretical discussion, this topic is also considered. Klem-Musatov (1980) derived a very comprehensive set of formulae for the diffracted amplitude due to a linear edge in a three dimensional medium. This work was the basis of the PhD. Thesis of Chan (1986) where the theory, along with numerous numerical experiments involving the comparison of the modified ART solution with “exact’’ solution methods, was presented.

A basic problem will first be considered to give a brief overview of the theory of edge diffractions. The geometry of this problem involves a wedge embedded in a halfspace and is such that both a source and receivers are located at the surface of the halfspace. Only reflected rays and primary diffractions are included in the solution. The motivation for this is to provide some initial insight for the extension of edge diffraction theory to more complicated and realistic geological structures. As an ART type solution is used, the geometrical and diffracted wavefields are computed separately and the total waveform is the sum of the two. This concept can be extended to combining a number of diffracting edge models and additively obtain the total waveform.

INTRODUCTION

The work of Klem-Musatov (1980) dealing with elastic waves diffracted by the linear edges of seismic interfaces, using an extension of asymptotic ray theory (ART) (Červený and Ravindra, 1970 and Červený, 2001, among others) and subsequently dynamic ray tracing (DRT) (Červený and Hron, 1980) were employed to obtain a high frequency approximation to elastodynamic waves diffracted by linear edges. The method presented uses modifications of ART and DRT incorporating the boundary layer method. This theory is applicable to a three dimensional case of rays, emanating from a point source, propagating in a geological model with homogeneous layers separated by curved interfaces. The Russian text by Klem-Musatov (1980), once relatively inaccessible may be found now in the English translation, Klem-Musatov (1995) as well as the translation of a more recent work, Klem-Musatov et al. (2007). A dissertation based on the original text (1980) is the topic of the Ph.D. thesis of Chan (1986), where many of the subjects contained in that book are discussed. The papers by Bakker (1990), Hron and Chan (1995) and Gallop and Hron (1997) pursue some aspects of the theory presented in the works of Klem-Musatov. The work of Bakker (1990) approaches the edge diffraction problem using a paraxial (DRT) approximation similar to that in Červený and Hron (1980) while Hron and Chan (1995) arrive at essentially the same results using a more conventional ART modified by the inclusion of boundary layer theory. As stated above, it is the intent of this paper to engage in a minimal amount of theoretical development and employ previously derived formulae to introduce concepts in a form intended for numerical implementation.

Formulae for use in the computation of numerical results would ideally be such that they would (a) allow for simple physical interpretation, (b) provide a practical means for the efficient calculation of the diffracted field and (c) in light of (b), also maintain a reasonable degree of accuracy when describing the diffracted wavefield. A relatively inaccessible report by Hron and Covey (1988) contains a more programmer friendly treatment of edge diffraction theory. The ART approximation to the edge diffracted arrival consists of the zero order asymptotic expression for the incident wavefield at the diffractor, due to a point source, multiplied by what was termed a diffraction coefficient. The resulting diffracted wavefield was obtained by considering the diffractor as a secondary source of seismic energy and tracing its path to receivers in accordance with a modified form of Snell’s Law and zero order ART. If the model were truly three dimensional, the individual points of diffraction must be replaced by diffraction edges. The theory employed here may be referred to as 2.5 D due to the fact that out of plane geometrical spreading is included.

The accuracy of a method similar in development to the one described above was determined in the work of Chan (1986) where an approximate method for SH waves was compared.

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with the highly numerically accurate results obtained by the Alekseev Mikhailenko method (AMM), a pseudo-spectral method (see for example, Mikhailenko, 1985 and 1988).

As the treatment of this problem contained in Klem-Musatov (1995 and 2007) and Chan (1986) are quite comprehensive, minimal theoretical aspects of the problem will be dealt with in this work. For clarity, the problem will be defined in the next section along with some comments and a restatement of the final formulae required for computation of the diffracted wavefield due to an edge diffraction will be presented. Other rather inaccessible notes (Hron, 1986) might be of some use for those interested in the theoretical development of this problem but these are fairly mathematically intense and of minimal attraction to the majority of readers.

The two models that will be considered in this work are shown in Figures 1 and 2. They each consist of a wedge that is infinite in the $y$ direction with ray propagation constrained to the $(r, z)$ plane. The receivers in both geometries are along a surface receiver profile. Model I will be briefly considered first, and then used, with minor modifications, for Model II. Numerical results for the combination of Models I and II, termed Model III, are also presented.

**THEORY**

Consider a three dimensional Cartesian coordinate system $(r, y, z)$ in a halfspace $z > 0$ with an explosive point source of compressional $(P)$ waves located at the origin, (Figure 1). A semi-infinite wedge is assumed to be located at a depth $z_D$ below the surface, occupying the three dimensional space

$$(z_D \leq z < \infty, 0 \leq r \leq r_D, -\infty < y < \infty).$$

Receivers are placed at the surface in the $(r, z)$ plane containing the source. For this case the reflected rays from the top of the wedge and the diffracted $P$ wave from the edge at $D = (r_D, z_D)$ are the only arrivals considered in the resulting synthetic seismograms. The compressional $(P)$ wave velocities in the half plane, $\alpha$, and in the wedge strip, $\alpha_r$, are chosen such that $\alpha < \alpha_r$. The shear wave $(S)$ velocities are defined by the relation $\beta = \alpha / \sqrt{3}$ and the densities are obtained from Gardner’s Law. This will

Figure 1. Geometry of Model I for a point source and a surface array of receivers. The wedge (dark gray) occupies the 3D space $(z_D \leq z < \infty, 0 \leq r \leq r_D, -\infty < y < \infty)$. The point of edge diffraction is in the $(r, z)$ plane at $(r_D, z_D)$. Ray propagation is assumed to lie in this plane. The shadow region $\Omega_S$ (light gray) for the ray reflected from the top of the wedge is defined by the boundary ray. Only reflected arrivals at a surface receiver, $r$, such that $(r \geq 2r_D)$ are seen on the synthetic traces. The diffracted arrival appears at all offsets.

Figure 2. Geometry of Model II for a point source and a surface array of receivers. The wedge (dark gray) occupies the 3D space $(r_D \leq r < \infty, z_D \leq z < \infty, -\infty < y < \infty)$. As in Model I, the point of edge diffraction is in the $(r, z)$ plane at $(r_D, z_D)$. In this model a reflected arrival off the flank of the wedge is not considered, only the ray reflected from the top of the wedge of interest here. The boundary ray, which separates the illuminated, $\Omega_I$, region from the shadow, $\Omega_S$ (light gray) region is shown. In this instance, reflected arrivals from the top of the wedge are only seen at a surface receiver, $r$, such that $(r \geq 2r_D)$ are seen on the synthetic traces. As for Model I the diffracted arrival is seen at all offsets.
be referred to as Model I. A modification of Model I will be
considered, denoted as Model II and shown schematically in
Figure 2. This second model also consists of a wedge embedded
in a halfspace with the wedge being in the three dimensional
space \((z_D \leq z < \infty, r_D \leq r < \infty, -\infty < y < \infty)\). All other parameters are
similar to those for Model I, except for the elastic parameters in
the wedge, defined such that \(\alpha < \alpha_c\).

When considering diffraction from the wedge in Model I there
are two regions to be considered in the half plane, the illumina-
ted \((\Omega_i)\) and the shadow \((\Omega_s)\) regions (Figure 1). The illumina-
ted and shadow regions are separated from one another by
what is termed the boundary ray. In the simple situations being
considered here the reflected arrival only appears on the
synthetic traces in the illuminated regions while the diffracted
arrival exists in both regions.

The Fourier time transformed vector particle displacem ent of
an incident compressional wave generated at the point source
at the origin and recorded at the surface receiver arrays may be
written in terms of the zero order ART approximation as
\[
U_C(r, \omega) = \frac{F(\omega)\Pi}{L_C(r)} \exp[i\omega \tau_C(r)] C_S
\]
where \(\omega\) is the circular frequency, \(\tau_C(\mathbf{r})\) \([\tau_C(M)]\) is the travel time along the reflected ray from the source to a reflection
point on the wedge top, to some point \(\mathbf{r}\) defining a surface
receiver, \(L_C(\mathbf{r}) [L_C(M)]\), is the 3D geometric spreading of the ray
between the source and the point \(\mathbf{r}\) which for the reflected
arrival is given by
\[
\frac{1}{L_G(M)} = \left[\frac{\tau_C^2 + (2z_D)^2}{\tau_M^2 + (2z_D)^2}\right]^{1/2}. \tag{2}
\]
The reflected ray arrival time may be written
\[
\tau_G(M) = \left[\frac{\tau_C^2 + (2z_D)^2}{\tau_M^2 + (2z_D)^2}\right]^{1/2}. \tag{3}
\]

The term \(\Pi\) in equation (1) is the product of all reflection and
transmission coefficients encountered along the geometrical ray
from source to receiver. In the reflected ray case, \(\Pi\) is the elastic
PP reflection coefficient from the top of the wedge. The vector
\(C_S = [C_S^{(p)}, C_S^{(s)}]^T\) is the surface conversion coefficient vector
which partitions the incident particle displacement at the free surface receiver
into its constituent vertical and hori-
zontal components. \(F(\omega)\) is the Fourier
time transform of the band limited
source wavelet, \(f(t)\), \(t\) being time. A
spherically symmetric radiation pattern of this source function is assumed.

For the diffracted arrival the travel time from source to receiver consists of two
parts; the time it takes the ray to travel
form the source to the diffraction edge in the \((r,z)\) plane at \((r_D,z_D)\) plus the time
taken for the ray to progress from the
diffraction point \((D)\) to the receiver at
\(M\). This infers that a diffracted arrival
generally has a different ray parameter
(horizontal component of the slowness vector) on either side of the edge
diffraction point.

From field data observations and results
obtained from numerical modeling with
finite difference methods a number of
attributes of edge diffracted arrivals
may be realized: (a) the diffracted wave-
field is frequency dependent and (b)
angular dependent measured at some
unit distance about the diffraction point
on the diffracting edge and (c) the point
on the diffracting edge appears to act as
a secondary source. For this reason a
solution is assumed that may be written
in terms of the reflected arrival at some
minimal distance about the point \(D\). The
diffracted arrival solution results by
multiplying the reflected arrival at the

Figure 3. Analytic continuation of the geometrical wavefield into the shadow region to obtain some simu-
lated measure of the geometrical arrival time in this region. A more complete discussion may be found in
Appendix A.
point \( D \) by a radiation characteristic function, \( l(\omega, \psi) \). This function is assumed to be a frequency \( (\omega) \) and angular \( (\psi) \) dependent in the \((r,z)\) plane. The actual determination of this function is yet to be addressed. However, based on empirical observations, the diffracted arrival may be written as

\[
U_D(r, \omega) = \frac{F(\omega)l(\omega, \psi)\Pi}{L_D(r)} \exp[i\omega\tau_D(r)]C_S. \tag{4}
\]

The three dimensional geometrical spreading \( L_D(r) = L_D(M) \) is the sum of the spreading from the source to the point of diffraction plus the addition of the spreading from this point to \( M \). This quantity and the diffracted travel time will be given later.

It is convenient at this point to introduce another Cartesian coordinate system, \((u,v)\), whose origin is at \( D \), together with a related polar coordinate system, \((r,\psi)\). Here, \( \psi \) is positive/negative in the shadow/illuminated regions which are defined by the boundary ray. The quantity \( \rho_M \) is the distance from the point of diffraction at \( D \) to the observation point \( M \) fully defined by the additional coordinate, \( \psi_M \) within the context of the Cartesian system \((u,v)\).

In light of the above Cartesian and polar coordinate systems definitions, the radiation characteristic function \( R(\omega, \psi) \) will be replaced in Equation (4) by a more general related function, \( W(z) \) with \( z = z(\omega, \omega, \rho_M, \psi_M) \), termed the diffraction coefficient. This coefficient is derived in a number of the previously cited works to which interested readers are referred as its inclusion here is, apart from introducing unnecessary length, is dealt with quite rigorously in the citations listed in the Introduction.

The vector particle displacement at the point \( M \), which is contained in the surface receiver array at the coordinates \((\rho_M, \psi_M)\) may then be written as

\[
U_D(M, \omega) = \frac{F(\omega)W(z)\Pi}{L_D(M)} \exp[i\omega\tau_D(M)]C_S. \tag{5}
\]

where \( z = z(\omega, \omega_M) \) and \( \omega_M = \omega(\omega, \rho_M, \psi_M) \), with \( \omega \) being the circular frequency and

\[
W(z) = \pm e^{-z^2} \frac{\sin(\pi/2 z \omega)}{\sqrt{\pi}} \tag{6}
\]

with the “−” and “+” signs associated respectively with the illuminated and shadow regions, and \( W(z) \) is the scaled complementary error function. The asymptotic expansion for \( W(z) \) for large values of \( |z| \), \( |\arg(z)| < \pi/4 \), is

\[
W(z) = i/\sqrt{\pi} \left( e^{i\pi/4} \frac{(\pi/2 z \omega)}{\sqrt{\pi}} \right) = e^{i\pi/4} \sqrt{\omega} (\tau_D - \tau_R), \tag{7}
\]

indicating that the amplitude of the diffracted arrival is of the order \( O(1/\sqrt{\omega}) \) for large values of the argument of \( z \). The meaning of the quantity \( \omega \) is found by studying the works dealing with solution method for edge diffracted waves and is defined as

\[
w^2 = \begin{cases} (a). \frac{2\omega}{\pi} (\tau_D(M) - \tau_C(M)) & \text{illuminated zone} \\ (b). \frac{2\omega}{\pi} \left( \frac{\rho_M}{\alpha} (1 - \cos \psi_M) \right) & \text{shadow zone} \end{cases} \tag{8}
\]

Equation (8.b) for \( w^2 \) results from the fact that at points at the surface no reflected arrival exists in the shadow zone \((r_D < r < \infty)\). Equation (8.b) was obtained using analytic continuation. Appendix A contains a discussion of its derivation in a simplified manner avoiding excessive mathematical rigor. Figure 3 depicts schematically what is discussed in Appendix A.

The geometrical spreading \( L_D(M) \) and arrival time \( \tau_D(M) \) for the diffracted \( P \) arrival for Model I may now be written as

\[
L_D(M) = \left[ \frac{\rho_M^2 + r_D^2}{\alpha^2} \right]^{1/2} + \rho_M \tag{9}
\]

\[
\tau_D(M) = \frac{\rho_M^2 + r_D^2}{\alpha^2} + \frac{\rho_M}{\alpha} \tag{10}
\]

respectively. As before, the compressional wave velocity in the halfspace, except for the wedge, is \( \alpha \). All other quantities have been previously defined.

Another simple geological model, depicted schematically in Figure 2 and denoted as Model II, will now be examined. It is similar to Model I with the exception that the wedge now occupies the space \((r_D < r < \infty, z_D < z < \infty, y < \infty)\) in the halfplane \((z > 0)\). The expressions for the reflected and diffracted waves amplitudes and travel times are given by Equations (1) and (5), and (3) and (10) still hold, provided the coordinate system \((\rho, \psi)\) is properly defined.

An additional model is constructed by overlaying Models I and II, and referred to as Model III. Numerical results will be presented for this situation.

**NUMERICAL RESULTS**

The geological parameters defining the models mentioned earlier in this report are given in Table I. Additional quantities required for model definition are the distance from the surface to the tops of the wedges, \( h = 400m \), the horizontal distance from the source location to the points of diffraction, \( r_D = 1000m \) and the horizontal distance from the source to the points the boundary ray in Models I and II where surface receivers are both at \( r_D = 2000m \). The offsets of all of the synthetic seismograms presented run from 0.0m to 3000m in steps of 50m. These together with a time scale and a brief description of what is shown in a given figure appear on all of the synthetic seismograms. A Gabor wavelet is used when producing the synthetic traces. The function \( W(z) \) is a standard function available in most mathematical libraries such as IMSL, where it is denoted as ZERFE (Gautschi, 1979a and 1979b).

Schematics of the models used are given in Figures 1 and 2 where additional information may be found in the captions. The synthetic seismograms presented include both the vertical and horizontal components of displacement at the surface for an elastic halfspace with an embedded wedge or wedges. Large velocity contrasts between the halfspace and the wedges were employed to emphasize the reflected arrivals.

Figures 4 through 7 are associated with Models I and II with Figures 8.a and 8.b displaying the vertical and horizontal components of displacement of Model III – a combination of Models I and II. There are 3 panels in each of the Figures 4 – 7 showing: (a) the reflected arrival, (b) the diffracted arrival and
Figure 4. Vertical component of displacements of the (a) direct, (b) diffracted and (c) combined, for Model I with receivers located at the model surface.
Figure 5. Horizontal component of displacements of the (a) direct, (b) diffracted and (c) combined, for with receivers located at the model surface.
Figure 6. Vertical component of displacements of the (a) direct, (b) diffracted and (c) combined, for Model II with receivers located at the model surface.
Figure 7. Horizontal component of displacements of the (a) direct, (b) diffracted and (c) combined, for Model II with receivers located at the model surface.
The models were designed to investigate properties of edge diffraction from a 3D wedge. Specifically, the introduction of the boundary ray for different geometries and ray types were considered. This ray is such that it defines the illuminated and shadow zones for the reflected arrival. The diffracted ray exists in both regions while the reflected ray exists only in the illuminated region. The formulae for the edge diffracted arrival were obtained from other works cited in the Introduction. The smooth transition of the diffracted arrival across the boundary ray from the illuminated region to the shadow region was taken as an indication that the formulae being used satisfied this constraint. Comparison of the modified ART solution for this problem has been checked for more complex media using finite difference and related methods and as such has not been included here. The diffraction coefficient is a function of the difference of the diffracted and direct travel times. As the direct arrival does not exist in the shadow zone it was necessary to introduce the concept of analytic continuation to provide an appropriate value for this quantity. What has been presented here is not a definitive source of all theory that is required for the introduction of edge diffracted arrivals into synthetic traces for complex structures, but rather a simple introduction to the topic to provide a basis for the numerical implementation of diffraction theory.

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Figure 8. Vertical (a) and horizontal (b) components of displacement for the combination of reflected and diffracted arrivals, for Models I and II with receivers located at the model surface. The displacement of the two models is just the additive sum of the constituent parts.
**APPENDIX A**

Assume a point source of compressional ($P$) waves located at the origin of an isotropic 3 dimensional Cartesian half space in which there is embedded a wedge of infinite dimension in the $y$ direction (Figure 3). At a time after an impulsive excitation of the point source the direct wavefront will have progressed to define the wavefront surface at $\tau_G$. Wavefronts resulting from reflection from the top of the wedge and transmission through it have not been considered. However, the diffracted wavefront from the impinging of the wavefront originating at the origin on point $D$ is included. This diffracted wavefront defined by $\tau_D$ will be assumed to be propagating in the half-space. The ray associated with the point source at the origin which is such that it passes some small distance $\varepsilon : \varepsilon \to 0$ from the point $D$ will be called the boundary ray and denoted $R_B$. This ray separates the illuminated zone ($\Omega_I$) from the shadow ($\Omega_S$) zone for the direct wavefront which originates from a point source at some arbitrary origin, $O$.

In the shadow zone the direct geometrical arrival does not exist even though its travel time in this region is required to determine the argument of the diffraction coefficient, $W(z)$, where

$$z = e^{j\pi/4} \sqrt{2/\omega} \; w \quad (A.1)$$

and $w$ is defined (tentatively) by the relation

$$w^2 = \left[ \frac{2\omega}{\pi} (\tau_D(M) - \tau_M(M)) \right]. \quad (A.2)$$

In the shadow region the travel time of the geometrical arrival to the point $M$ must be determined by analytic continuation of $\tau_G(M_0)$ from the shadow/illuminated boundary into the shadow region. As $S_T$ is the tangent plane to $\tau_G(M_0)$ on the boundary ray $R_B$ it may be interpreted as the local representation of $\tau_G(M_0)$ there, which is to say that from the view point of seismic energy partitioning due to encounters of the ray with interfaces it is identical to a seismic plane wave at that point. From a mathematical point of view the analytic continuation of the travel time along the plane representation of the wave front $S_T$ requires that the travel time is the same along the plane wave front as it is at $M_0$. The same argument may be used to infer that the amplitude along the plane wave front $S_T$ must also be the same as its value at $M_0$.

Using the coordinate system $(\rho, \psi)$ defined in the text which is centered at the diffraction point $D$, assuming that all rays are constrained to lie in the $(x,z)$ plane of the $(x,y,z)$ Cartesian system initially assumed here, the travel time of the direct geometrical arrival on the seismic boundary ray, is and as a result of the preceding argument

$$\tau_G(M) = \tau_G(M_0) \quad \text{where} \quad M \in S_T. \quad (A.3)$$

If $\tau_D$ is the time taken for the direct geometrical arrival to travel from the point source to the edge diffraction point $D$, then it may be seen from Figure 3 that

$$\tau_D(M) = \tau_1 + \frac{\rho_M}{\alpha} \quad (A.4)$$

and

$$\tau_G(M) = \tau_1 + \frac{\rho_M \cos \psi_M}{\alpha} \quad (A.5)$$

where $\alpha$ was defined in the text as the wave velocity in the half space. Substituting equations (A.4) and (A.5) into (A.2) the quantity $z$ expressed in terms of $w$ may be obtained from

$$w^2 = \left[ \frac{2\omega}{\pi} (\tau_D(M) - \tau_G(M)) \right] = \left[ (\frac{2\omega}{\pi}) \left( \frac{\rho_M}{\alpha} \right) (1 - \cos \psi_M) \right]. \quad (A.6)$$