CSEG JOURNAL, V. 8, NO. 1 (DEC., 1972) P. 14-21, 5 FIGS.

THE FOURIER SPECTRUM OF GRAVITY ANOMALIES DUE TO TWO-DIMENSIONAL PRISMS†

T. W. F. EBY*

A mathematical basis for the application of Fourier spectrum analysis to Bouguer gravity profile interpretation is developed. Isolated two-dimensional prisms, and small assemblages of prisms of varying thickness, width, depth, and of uniform density, are used as models to simulate geologic bodies. The influences of the geometric parameters on the shape of the Fourier spectrum is assessed. A method of estimating depth to source, and source thickness, by analytic continuation of the shift in the frequency domain, and measurement of the shift in the frequency position of the maxima of the spectrum curve is presented. It is shown that this method should be applicable, at least for depth estimation, both to isolated prisms and to small assemblages of prisms.

INTRODUCTION

The application of Fourier analysis to the interpretation of potential field data can be traced back at least ten years, (see, for example, Troshkov and Shalaev, 1961). During this time a number of investigators, (Bhattacharyya, 1964 and 1966); Spector and Bhattacharyya, 1966); Spector, 1968; and Spector and Grant, 1970), have investigated the application of this technique to aeromagnetic data. Because of the usually large extent of aeromagnetic surveys, the interpretational emphasis has been placed on statistical models and ensemble averaging.

However, because of the relatively small areas included in many gravity surveys, it is often unrealistic to assume a normal distribution of gravity source parameters. A need exists for an improved understanding of the behaviour, and interpretation, of small assemblages of monopolar sources. Davis (1971) has shown that the characteristics of the amplitude spectrum of two dimensional gravity fault anomalies can be used, in conjunction with band pass filters, to derive source parameters. In this paper I discuss the form of the amplitude spectrum of the gravity anomaly produced by two-dimensional prisms, and present an interpretational technique for extracting the source parameters from the spectrum. The method suggested can be extended to the three-dimensional case, and to other simple geometries in a straight-forward manner.

THE SPECTRUM OF GEOMETRIC SOURCES

The expression for the Fourier amplitude spectrum of the field of an arbitrary two-dimensional body can be written in the form:

$$T(k) = KO(\Theta)D(k)G(k,\alpha,\beta,\sigma),$$

where: k = *the angular wave number*

- K is a factor dependent on the mass of the body
- $O(\theta)$ is a factor dependent on the orientation of the field
- D(k) depends on the depth to the causitive body
- $G(\alpha,\beta,\sigma,k)$ depends on the particular geometry of the source, and
- (α,β,σ) are used here to indicate the geometric parameters.

†Manuscript received by Editor October 11, 1972; revised manuscript received November 14, 1972.

*Amoco Canada Petroleum Company Ltd.

I assume that K does not vary with position and is, therefore, a constant determined by the mass of the source.

The factor $0(\theta)$ can be assumed to be unity in the case of the gravity field (see, Cameron and Eby, 1971).

Therefore, the form of the spectrum will be determined by the two factors D(k) and $G(k, \alpha, \beta, \sigma)$.

ISOLATED TWO-DIMENSIONAL PRISMS

I assume that the sources are infinite in the direction at right angles to the profile line defining the gravity anomaly.

Proceeding in a similar manner to Spector (1968), the amplitude spectrum of a two-dimensional monopolar prism is given by:

$$T_{prism}(k) = \frac{2\pi K}{k} \frac{\sin(ka)}{(ka)} e^{-hk} (1-e^{-tk}),$$

where: a = the width of the prism
t = the thickness of the prism
h = the depth to the top of the prism
k and
K have been defined previously.

We may write a modified, (see Cameron and Eby, 1971), normalized amplitude spectrum as:

$$kT(k) = \frac{\sin(ka)}{(ka)} e^{-hk}(1-e^{-tk}) \qquad (1)$$

Case 1: For small a, $\sin(ka)/(ka) = 1$

In this case, the form of the modified spectrum, defined in equation (1), is primarily determined by the expression:

$$L(k) = e^{-hk} (1-e^{-tk}). \qquad (2)$$

Figure 1(a) illustrates this function for t=500 feet, and values of h as indicated on the figure. Figure 1(b) shows the result for t=1000 feet and h as indicated. The ordinate in these figures is given in units of f, where $f = k/2\pi$. It is evident from these figures that an attempt to estimate source depth from the slope of the curves, as has been suggested by Spector (1968) for the analysis of aeromagnetic data, would result in an underestimation of the true depth.

Setting the thickness to depth ratio, $t/h = \gamma$, in equation (2) and calculating

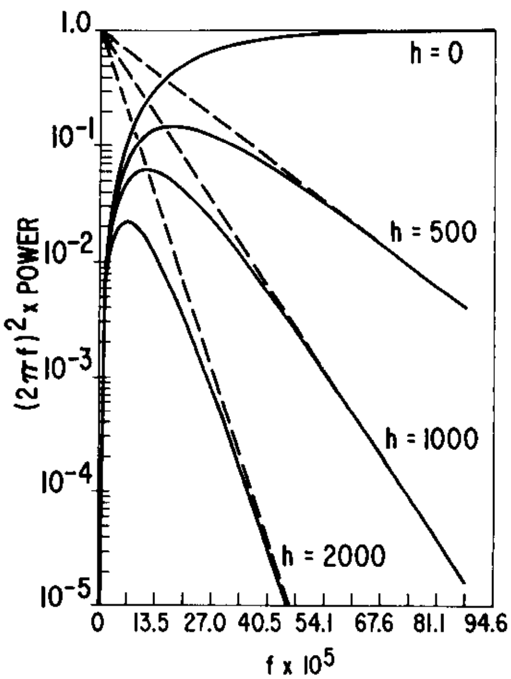

Fig. 1(a). Power spectrum curves for t = 500 and h as shown on the curves. The broken line shows the function D(f) = $e^{-4\pi fh}$ for values of h corresponding with the L(f) values.

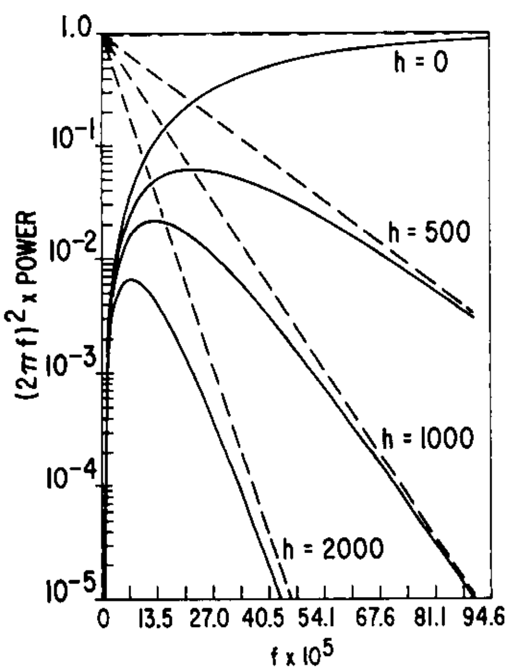

Fig. 1(b). Power spectrum for t = 1000 and h as shown on the curves.

the position of the maximum of this function in the usual manner yields:

$$k_{max} = \frac{1}{h}\,\frac{\ln(\gamma+1)}{\gamma}$$

or
$$f_{max} = \frac{1}{2\pi h}\,\frac{\ln(\gamma+1)}{\gamma} \; . \qquad (3)$$

A plot of f_{max} against γ, with h as the parameter, is shown in figure 2.

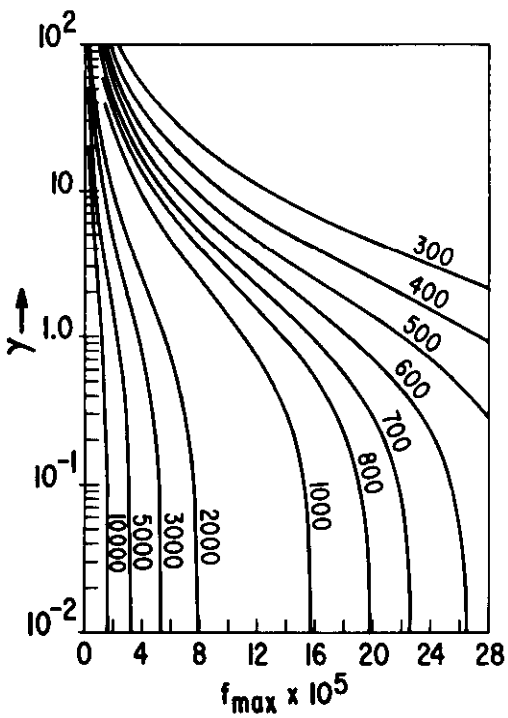

Fig. 2. A plot of f_{max} against γ with h as parameter as shown on curves.

If we can locate the position of f_{max} on a plot of the modified amplitude spectrum of the field, we may use equations (3) or the curves of figure 2 to tabulate corresponding estimates of γ and h. Figure 2 indicates that the change in f_{max} with h is greatest when h is of the order of a few hundred to a few thousand feet, and γ lies in the range $\gamma = .1$ to $\gamma = 10.0$.

In an attempt to remove the γh ambiguity, we examine the differential:

$$\partial f_{max} = -\frac{\ln(\gamma+1)}{2\pi\gamma} \cdot \frac{1}{h^2}\,\partial h$$

Making the approximations

$$\partial f_{max} = \Delta f_{max} \text{ and } \partial h = \Delta h$$

we may write:

$$h = -\frac{f_{max}\,\Delta h}{\Delta f_{max}} \; . \qquad (4)$$

Δf_{max} may be calculated by assuming some value for Δh and continuing the field upwards or downwards by Δh (ie: by multiplying the spectrum by $e^{\pm 2\pi \Delta h f}$) and measuring the resulting displacement in the position of f_{max}.

The approximation used to obtain equation (4) assumes that $\frac{\partial f_{max}}{\partial h}$ does not vary appreciably with f_{max}. In general, this assumption is not valid.

Let us assign the coordinates (f_{max1}, h_1) to the point representing the maximum position of the original spectrum; and (f_{max2}, h_2) to the point representing the maximum of the spectrum after continuation. Then $h = h_2 - h_1$, and $f_{max} = f_{max2} - f_{max1}$. Now a simple application of the "law of the mean" yields the exact expressions:

$$h_1 = \frac{-\Delta h\, f_{max1}}{\Delta f_{max}} \qquad (5)$$

$$\text{or} \quad h_1 = \frac{\Delta h\, f_{max2}}{\Delta f_{max}}$$

Where the physically meaningful value of h_1 in equations (5) is the value accepted. This value h_1 can then be used in conjunction with the curves of Figure (2) to find γ. By differentiating h_1 with repect to f_{max1}, and to f_{max2}, we may derive the results

$$\frac{\delta h_1}{h_1} = \frac{f_{max2}\,\delta f_{max1} + f_{max1}\,\delta f_{max2}}{f_{max2}\,\Delta f_{max}}$$

and (6)

$$\frac{\delta h_1}{h_1} = \frac{f_{max2}\,\delta f_{max1} + f_{max1}\,\delta f_{max2}}{f_{max1}\,\Delta f_{max}}$$

where: δh_1 is the error in h_1,

δf_{max} is the error in the f_{max} value measured, and the first expression in (6) is associated with the first expression in (5); and the second expression in (6) is associated with the second expression in (5).

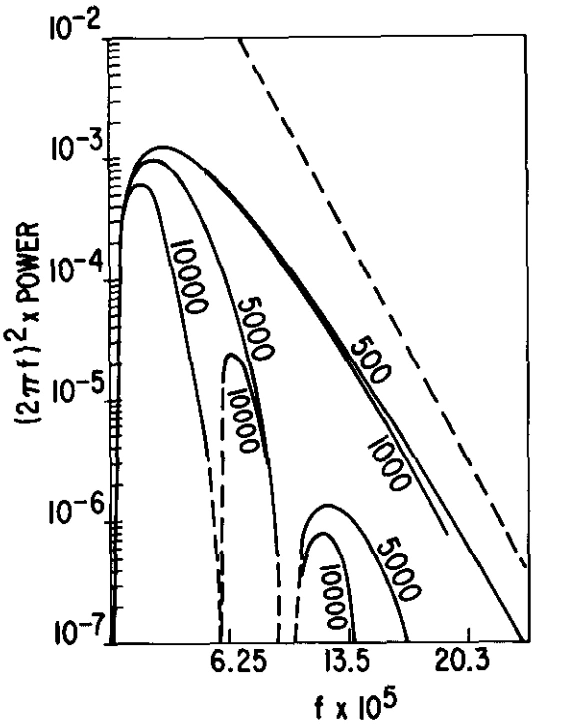

Fig. 3. The effect of the prism width on the form of the spectrum for h = 5000 feet and t = 500 feet and paramter 'a' as shown on curves.

Expressions (6) indicate that the error in h_t is controlled by Δf_{max}. Since Δf_{max} increases as Δh increases, Δh should be chosen to be large.

Case 2: $\sin(ak)/(ak) \neq 1$

The effect of the "geometry factor", $\sin(ak)/(ak)$ on the spectrum is illustrated in Figure 3. We see that this factor can seriously distort the function given in equation (2), and shift the position of f_{max}. However, since "a" can be estimated either from the position of successive null points in the spectrum, or from the anomaly half-widths, we may multiply the amplitude spectrum by $(ak)/\sin(ak)$ to remove the effect of geometry. The analysis then follows as given above.

THE SPECTRA OF SMALL ASSEMBLAGES OF SOURCES

Difficulties arise in the treatment of real geophysical data for two reasons:

1) Real sources, and their associated gravity anomalies, are seldom isolated.
2) A profile length longer than that defining one gravity anomaly may be necessary to give sufficient spectral resolution to make use of the technique described above.

For these reasons, it is necessary to consider the spectra produced by the field of two or more sources.

Since the Fourier transform is a linear operation, we may calculate the spectrum of an assemblage of sources by the addition of the spectra of the individual sources in the assemblage.

THE EFFECT OF WIDTH VARIATION

If the assemblage consists of two-dimensional prisms of identical thickness, and at identical depth, but with differing widths, a_j; equation (1) becomes:

$$kT(k) = e^{-hk}(1-e^{-tk}) \sum_{j=1}^{n} \frac{\sin(a_j k)}{(a_j k)} \quad (7)$$

Equation (7) is susceptible to the analysis described earlier in this paper if we estimate the half-widths of the sources and multiply the spectrum by the factor:

$$\left\{ \sum_{j=1}^{n} \frac{\sin(a_j k)}{(a_j k)} \right\}^{-1}$$

EFFECT OF THICKNESS VARIATION

I assume that the width and depth of the sources does not vary between prisms.

The modified amplitude spectrum may be written:

$$kT(k) = e^{-hk} \frac{\sin(ak)}{(ak)} \sum_{j=1}^{n} (1-e^{-t_j k}) \quad (8)$$

Setting $t_j = \bar{t} + \Delta t_j$

where: $\bar{t} = \frac{1}{n}\sum_{j=1}^{n} t_j$, the mean thickness of the sources, and Δt_j = the deviation in thickness of each source from the mean. Correcting for the effect of width, equation (8) becomes:

$$L(k) = e^{-hk}(n - e^{-\bar{t}k} \sum_{j=1}^{n} e^{-\Delta t_j k}) \quad (9)$$

For convenience we set:

$$\bar{\gamma} = \frac{\bar{t}}{h}$$

$$\gamma_j = \frac{t_j}{h}$$

and

$$\varepsilon_j = \frac{\gamma_j - \bar{\gamma}}{\bar{\gamma}} \qquad (10)$$

Then

$$L(k) = e^{-hk}\left(n - e^{-\gamma hk} \sum_{j=1}^{n} e^{-\gamma h \varepsilon_j k}\right)$$

Two body case: $n = 2$

For $n = 2$, $\varepsilon_1 = -\varepsilon_2$ and equation (11) reduces to:

$$L(k) = e^{-hk}\left(1 - e^{-\bar{\gamma}hk} \cosh(k\varepsilon\bar{\gamma}h)\right). \qquad (12)$$

Calculating the k position of the maximum of equation 12 gives:

$$[1 + \bar{\gamma}(1-\varepsilon)]\, e^{-\bar{\gamma}hk_{max}(1-\varepsilon)} + [1 + \bar{\gamma}(1+\varepsilon)]\, e^{-\bar{\gamma}hk_{max}(1+\varepsilon)} - 2 = 0 \qquad (13)$$

We may solve for k_{max} in equation 13 numerically. Figure 4 shows the results of this numerical solution for $h = 100$ feet and for ranges of γ and ε as is shown on the figure.

Differentiating equation 13 implicitly, yields:

$$2\bar{\gamma}e^{-\bar{\gamma}hk_{max}}\left(h\frac{dk_{max}}{dh} + k_{max}\right)$$

$$\{[1+\bar{\gamma}(1+2\varepsilon)] \cosh(\bar{\gamma}hk_{max}\varepsilon) - \varepsilon(1+2\bar{\gamma}) \sinh(\bar{\gamma}hk_{max}\varepsilon)\} = 0.$$

Thus:

$$h\frac{dk_{max}}{dh} + k_{max} = 0 \qquad (14)$$

if, and only if:

$$[1+\bar{\gamma}(1+2\varepsilon)] \cosh(\bar{\gamma}hk_{max}\varepsilon) \neq \varepsilon(1+2\bar{\gamma}) \sinh(\bar{\gamma}hk_{max}\varepsilon). \qquad (15)$$

From the equation 14 we may make the approximation:

$$h = \frac{-k_{max}\,\Delta h}{\Delta k_{max}} \qquad (16)$$

Equation (15) is identical to equation (4) and the analysis given earlier applies. Thus, knowing h and f_{max}, we may use nomographs similar to figure 4 to estimate values of $\bar{\gamma}$ and ε.

If condition (15) does not hold, then

$$\tanh(\bar{\gamma}hkmax\varepsilon) = \frac{1+\bar{\gamma}(1+2\varepsilon)}{\varepsilon(1+2\bar{\gamma})} \qquad (17)$$

(17) is true if $\bar{t} \leq \Delta t\left(\frac{h}{h+\bar{t}}\right)$.

But since $\bar{t} > \Delta t$ for real cases, we see that condition (15) will hold for real cases.

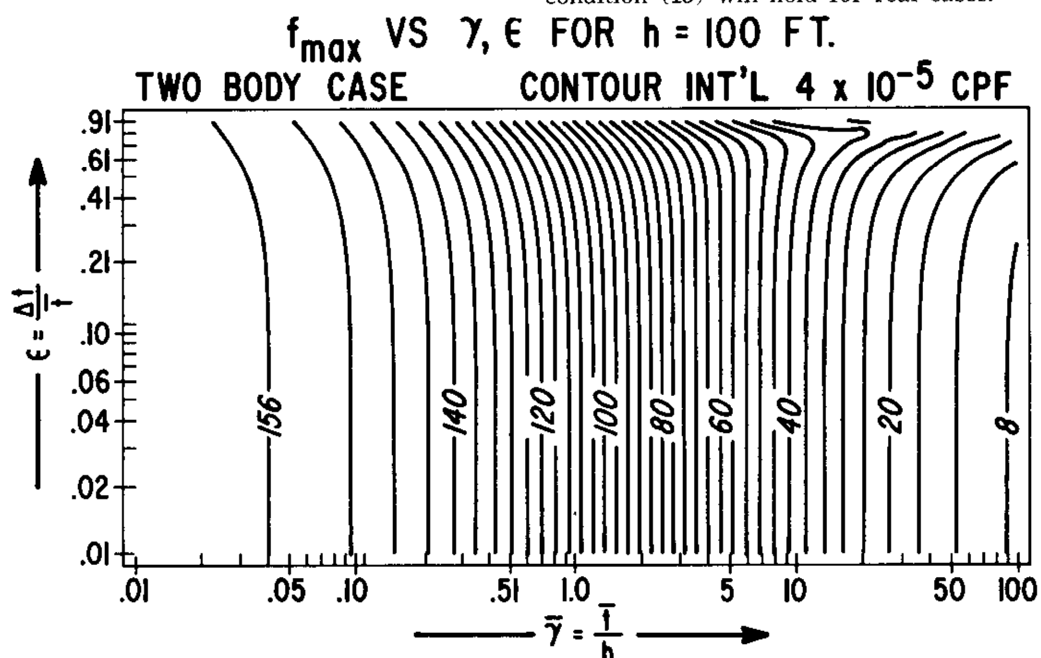

Fig. 4. Plot of f_{max} versus γ ε for $h = 100$ feet for two body case. Interval is 4×10^{-5} CPF.

Many body case: $n > 2$.

Expanding the summation in equation (9) in a power series and requiring that:

$$\sum_{j=1}^{n} (\Delta t_j)^3 \leq \frac{1}{k^3}$$

yields the following approximate expression for the modified amplitude spectrum

$$L(k) = e^{-hk}[1 - e^{-\bar{t}k}(1 + \frac{k^2}{2}\bar{\gamma}h^2\sigma_\varepsilon^2)] \quad (18)$$

where $\sigma_\varepsilon^2 = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\gamma_j - \bar{\gamma}}{\bar{\gamma}}\right)^2$,

and is a normalized measure of the variance in γ_j about the mean γ; and γ_j and γ are defined by relations (10).

Calculating the k position of the maximum of equation (18) yields:

$$e^{\bar{\gamma}hk_{max}} = (\bar{\gamma}+1) - k_{max}\, h\sigma_\varepsilon^2\bar{\gamma}^2 + \frac{(\bar{\gamma}+1)}{2} k^2_{max}\, h^2\, \sigma_\varepsilon^2\, \bar{\gamma}^2. \quad (19)$$

A numerical solution of this equation is shown in figure (5) for parameter values as indicated on the figure.

Calculating dk_{max}/dh from equation (19) gives:

$$\bar{\gamma}(k_{max} + h\frac{dk_{max}}{dh})(\sigma_\varepsilon^2\bar{\gamma} - (\bar{\gamma}+1)\sigma_\varepsilon^2\bar{\gamma}hk_{max} + e^{\bar{\gamma}hk_{max}}) = 0$$

and, again:

$$k_{max} + h\frac{dk_{max}}{dh} = 0$$

if, and only if:

$$e^{\bar{\gamma}hk_{max}} \neq (\bar{\gamma}+1)\sigma_\varepsilon^2\bar{\gamma}hk_{max} - \sigma_\varepsilon^2\bar{\gamma} \quad (20)$$

If condition (20) does not hold, then, using *equation (19)*:

$$k_{max} = \frac{1}{\bar{\gamma}h(\bar{\gamma}+1)}[(2\bar{\gamma}+1) \pm \frac{\sqrt{\sigma_\varepsilon^2 - 1}}{\sigma_\varepsilon}\sqrt{(\bar{\gamma}^2+1)+\bar{\gamma}^2}] \quad (21)$$

From (21) we see that k_{max} is complex if $\sigma_\varepsilon < 1$. Thus, provided that $\sigma_\varepsilon < 1$ there is no real k_{max} that does not satisfy condition (20). Therefore, provided that the standard deviation in thickness t_j about the mean thickness $\bar{t}$, is less than the mean $\bar{t}$, the suggested method of determining depth will hold.

THE EFFECT OF DEPTH OF VARIATION

Assume that the width and thickness of the sources are identical for all sources in the assemblage.

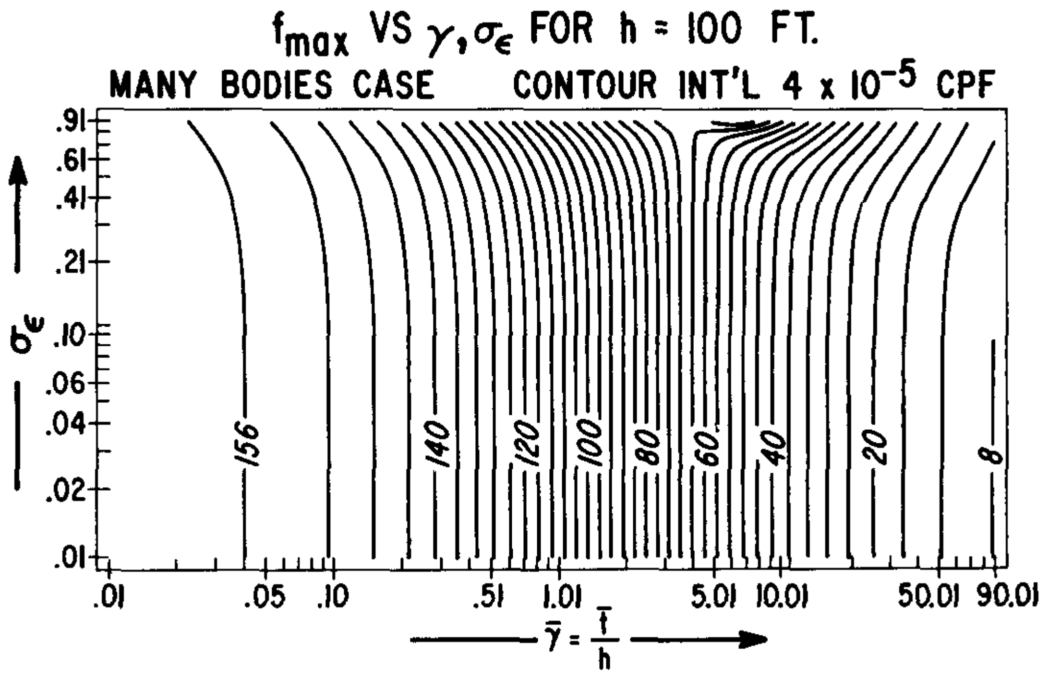

Fig. 5. Plot f_{max} versus γ and σ_ε for h = 100 feet for many bodies case. Interval is 4×10^{-5} CPF.

For n sources we may write

$$L(k) = (1-e^{-tk}) \sum_{j=1}^{n} e^{-h_j k} , \qquad (22)$$

setting $h_j = \bar{h} + \Delta h_j$,

where Δh_j is the variation in depth about the mean depth, $\bar{h}$:

Further, we set $\frac{\Delta h_j}{\bar{h}} = \zeta_j$

and $\qquad (23)$

$$\frac{t}{\bar{h}} = \alpha$$

Two body case: $n = 2$

Substituting relations (23) in equation (22) for $n = 2$, we find:

$$L(k) = (1-e^{-\alpha\bar{h}k})\, e^{-\bar{h}k}\, (e^{\zeta\bar{h}k} + e^{\zeta\bar{h}k}) \qquad (24)$$

Calulating the k position of the maximum of equation (24) yields:

$$(1-\zeta)e^{-\bar{h}k_{max}(1-\zeta)}+(1+\zeta)e^{-\bar{h}k_{max}(1+\zeta)} -(1+\alpha-\zeta)e^{-\bar{h}k_{max}(1+\alpha-\zeta)} -(1+\alpha+\zeta)e^{-\bar{h}k_{max}(1+\alpha+\zeta)} = 0 \qquad (25)$$

Equation (25) may be solved numerically for desired values of $\bar{h}$, α, and ζ. Calculating dk_{max}/dh in equation (25) yields:

$$e^{-\bar{h}k_{max}}[\bar{h}\frac{dk_{max}}{dh} + k_{max}]\{2\cosh(\bar{h}k_{max}\zeta) [(1+\zeta^2)-\{(1+\alpha)^2+\zeta^2\}e^{-\bar{h}k_{max}\alpha}] -4\zeta\sinh(\bar{h}k_{max}\zeta)[1-(1+\alpha)e^{-\bar{h}k_{max}\alpha}]\} = 0 .$$

Thus: $\bar{h}\frac{dk_{max}}{dh} + k_{max} = 0$

$$\qquad (26)$$

if, and only if: $\tanh(\bar{h}k_{max}\zeta)$

$$\neq \frac{[(1+\zeta)^2-\{(1+\alpha)^2+\zeta^2\}e^{-\bar{h}k_{max}\alpha}]}{2\zeta[1-(1+\alpha)e^{-\bar{h}k_{max}\alpha}]} \qquad (27)$$

And, provided that condition (27) holds, the average depth to source may be derived as previously described.

Condition (27) implies that difficulties may arise when $\alpha \simeq 0$ or when $h \simeq 0$. This situation could be anticipated of course, since in either case the maximum of equation (24) will occur at very large values of k_{max} and will be poorly defined.

Many Body Case: $n > 2$.

Proceeding in a manner similar to that used for the case of thickness variation, we find:

$$L(k) \simeq (1-e^{-\alpha\bar{h}k})e^{-\bar{h}k}(1+\frac{\bar{h}^2k^2}{2}\sigma_\zeta^2) \qquad (28)$$

where σ_ζ^2 is the variance in ζ_j.

Then, calculating the maximum equation (28) yields:

$$[1-\bar{h}\sigma_\zeta^2 k_{max}(1+\frac{\bar{h}k_{max}}{2})]-e^{\alpha\bar{h}k_{max}}$$

$$\{(1+\alpha)-\bar{h}k_{max}\sigma_\zeta^2[1+\frac{\bar{h}k_{max}}{2}(1+\alpha)]\} = 0 \qquad (29)$$

Again, this equation may be solved numerically.

Calculating dk_{max}/dh, as before, yields:

$$(k_{max}+\bar{h}\frac{dk_{max}}{d\bar{h}})\{\sigma_\zeta^2(1+\bar{h}k_{max})-e^{-\alpha\bar{h}k_{max}} [\sigma_\zeta^2(1+\bar{h}k_{max}(1+\alpha)) -\alpha\{(1+\alpha) -\bar{h}k_{max}\sigma_\zeta^2(1+\frac{\bar{h}k_{max}}{2}(1+\alpha))\}]\} = 0 .$$

Thus: $\bar{h}\frac{dk_{max}}{d\bar{h}} + k_{max} = 0$

if, and only if:

$$\sigma_\zeta^2(1+\bar{h}k_{max})-e^{-\alpha\bar{h}k_{max}}[\sigma_\zeta^2 (1+\bar{h}k_{max}(1+\alpha))-\alpha\{(1+\alpha)-\bar{h}k_{max}\sigma_\zeta^2 (1+\frac{\bar{h}\,k_{max}}{2}(1+\alpha))\}] \neq 0 . \qquad (30)$$

Again, provided that condition (30) holds, the average depth to source may be found in the normal manner.

It is readily seen that in cases of interest, i.e., σ_ζ^2 small, condition (30) will hold.

CONCLUSIONS

In cases where the source of a gravity anomaly can be approximated by a two-dimensional prism, it should be possible to use a modified form of the power or amplitude spectrum in conjunction with analytic continuation, to directly obtain source depths. It has been shown that the depth to the source is given by:

$$h = \frac{f_{max}\,\Delta h}{\Delta f_{max}} \qquad (31).$$

where: h — the depth to the top of the source

f_{max} = the position of the maximum value of the modified power or amplitude spectrum.

Δf_{max} = the shift in the position of the maximum after continuation by a distance Δh.

Further, it has been shown that this method should work for assemblages of two or more sources, even if the sources vary in thickness and in depth to source. In the later case, the depth given by relation (31) will be the mean depth to source.

ACKNOWLEDGEMENT

The author wishes to thank Amoco Canada for their permission to publish this paper.

REFERENCES

Bhattacharyya, B. K., 1966, Continuous spectrum of the total magnetic field anomaly due to a rectangular prismic body: Geophysics, v. 31, p. 97.

Bhattacharyya, B. K., 1964, Magnetic anomalies due to prism-shaped bodies with arbitrary polarization: Geophysics, v. 29, p. 517.

Cameron, J. B., and Eby, T. W. F., 1971, Introduction to gravity Exploration: Exploration Department, Amoco Canada Petroleum Company Ltd.

Davis, Thomas M., 1971, A filtering technique for interpreting the gravity anomaly generated by a two-dimensional fault: Geophysics, v. 36.

Spector and Bhattacharyya, B. K., 1966, Energy density spectrum and autocorrelation function of anomalies due to simple magnetic models: Geophys. Prosp., v. 14, p. 242.

Spector, A., 1968, Spectral analysis of aeromagnetic data: PhD. Thesis, Department of Physics, University of Toronto.

Spector, A., and Grant, F. S., 1970, Statistical models for interpreting aeromagnetic data: Geophysics, v. 35, no. 2.

Troshkov, G. A., and Shalaev, S. V., 1961, Application of the Fourier Transform to the solution of the reverse problem of gravity and magnetic surveys: Trans. by N. Rothenberg, Jour. Can. Soc. Expl. Geophys., April 1968, p. 46-62.