INTERPRETATION OF THE ANOMALOUS MAGNETIC FIELDS GENERATED BY A UNIFORMLY MAGNETIZED SPHERE USING FOURIER INTEGRALS†

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ABSTRACT

The present investigation seeks to interpret the magnetic anomalies generated by a uniformly magnetized buried sphere, using Fourier Integrals. As a first step, the horizontal and vertical components of the theoretical magnetic anomaly due to a sphere are represented in the frequency domain. Then, analyses of these transforms are made in an attempt to determine uniquely as many parameters relating to the buried sphere as possible. It is shown that the depth to the centre of the sphere can be uniquely determined from the amplitude spectrum and that an idea of the size of the sphere may also be obtained. It is also shown that improved resolution is attained by spectral analysis in the case of two spheres occurring together.

INTRODUCTION

In recent years many investigators (Odegard and Berg (1965), Bhattacharya (1966), Gudmundsson (1966), Spector and Bhattacharya (1966), Sharma and Geldart (1968), Sharma and Gill (1970), Bhattacharya (1971), Eby (1972), Rae and Avasthi (1973), Sengupta (1975 a,b), Sengupta and Das (1975)) have presented analyses of potential data in the frequency domain. Most of these works are on theoretical spectra of different geometrically shaped bodies.

Spectral analysis of potential data may be classified broadly into two groups. In the first group, spectral analyses are carried out to design spatial filters for improving signal to noise ratios in field data. In the second group, Fourier spectra of the effects due to different types of bodies are investigated in order to determine the parameters related to a particular body. The present investigation falls into the second category. In this case, the Fourier transform of the anomalous magnetic field due to a uniformly magnetized sphere is obtained analytically. The depth to the centre of the body is determined from the amplitude spectrum.

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List of Symbols

ω	-	Angular frequency in radians/Km.
ΔZ, ΔH	-	Vertical and horizontal components of the anomalous magnetic field due to a uniformly magnetized sphere.
$F_{\Delta Z}(\omega)$, $F_{\Delta H}(\omega)$	-	Fourier transforms of the vertical and horizontal components of the anomalous magnetic field.
Z_0, H_0	-	Vertical and horizontal components of the earth's magnetic field
k	-	Susceptibility contrast.
d	-	Depth to the centre of the sphere.
x	-	Distance of the point of observation from the origin.
R	-	Radius of the sphere.
$A(\omega)$	-	Amplitude spectrum
$\phi(\omega)$	-	Phase spectrum.
V	-	Volume of the sphere.

Fourier Transform of the Anomalous Field Due to a Sphere

The vertical (ΔZ) and horizontal (ΔH) components of the anomalous magnetic field due to a uniformly magnetized sphere are given by *Heiland (1951)*:

$$\Delta Z = -\frac{kV}{r^5}[H_0 \cdot 3xd + Z_0(x^2 - 2d^2)]$$

$$= -kV\left[\frac{H_0 \cdot 3xd}{(x^2+d^2)^{5/2}} + Z_0\left(\frac{1}{(x^2+d^2)^{3/2}} - \frac{3d^2}{(x^2+d^2)^{5/2}}\right)\right] \quad (1)$$

and

$$\Delta H = -\frac{kV}{r^5}[Z_0 \cdot 3xd - H_0(2x^2 - d^2)]$$

$$= -kV\left[\frac{Z_0 \cdot 3xd}{(x^2+d^2)^{5/2}} - H_0\left(\frac{2}{(x^2+d^2)^{3/2}} - \frac{3d^2}{(x^2+d^2)^{5/2}}\right)\right] \quad (2)$$

Erdeyli (1954) and Dwight (1960) have shown that

$$\int_0^\infty \frac{1}{(x^2+d^2)^{1/2}} \cos \omega x \, dx = K_0(\omega d) \quad (3)$$

and

$$\int_0^\infty \frac{x}{(x^2+d^2)^{3/2}} \sin \omega x \, dx = \omega K_0(\omega d) \quad (4)$$

where $K_o(\omega d)$ id the modified Bessel function of the 0 th order. Since both the integrations are with respect to x we can differentiate the integrands with respect to d.

$$\int_0^\infty \frac{d}{(x^2+d^2)^{3/2}} \cos \omega x \, dx = \omega K_1(\omega d) \quad (5)$$

$$\int_0^\infty \frac{3xd}{(x^2+d^2)^{5/2}} \sin \omega x \, dx = \omega^2 K_1(\omega d) \quad (6)$$

where $K_1(\omega d)$ is the modified Bessel function of the first order. Differentiating (5) with respect to d again,

$$-\int_0^\infty \frac{3d^2}{(x^2+d^2)^{5/2}} \cos \omega x \, dx + \int_0^\infty \frac{1}{(x^2+d^2)^{3/2}} \cos \omega x \, dx$$

$$= -\omega^2\left[K_0(\omega d) + \frac{K_1(\omega d)}{\omega d}\right] \quad (7)$$

It is also known that the Fourier transform of an even function is equal to its cosine transform, and that of an odd function is equal to its sine transform. By using the equations (5), (6) and (7), Fourier transforms of the vertical and horizontal components of the anomalous fields, as expressed in equations (1) and (2), can be derived. These are:

$$F_{\Delta Z}(\omega) = 2kV\left[Z\omega^2\left(K_0(\omega d) + \frac{K_1(\omega d)}{\omega d}\right) + iH_0\left(\omega^2 K_1(\omega d)\right)\right] \quad (8)$$

$$F_{\Delta H}(\omega) = 2kV\left[iZ_0\left(\omega^2 K_1(\omega d)\right) - H_0\left(\omega^2 K_0(\omega d)\right)\right] \quad (9)$$

where $i = \sqrt{-1}$

The amplitude and phase spectra as obtained from equations (8) and (9) are as follows:

$$[A(\omega)]_{\Delta Z} = 2kV\omega^2\left[Z_0^2\left(K_0(\omega d) + \frac{K_1(\omega d)}{\omega d}\right)^2 + H_0^2\left(K_1(\omega d)\right)^2\right]^{1/2} \quad (10)$$

$$[A(\omega)]_{\Delta H} = 2kV\omega^2\left[Z^2\left(K_1(\omega d)\right) + H_0^2\left(K_0(\omega d)\right)^2\right]^{1/2} \quad (11)$$

$$[\phi(\omega)]_{\Delta Z} = \text{Arctan}\left[\frac{K_1(\omega d)}{K_0(\omega d) + K_1(\omega d)/\omega d} \quad \frac{H_0}{Z_0}\right] + 2n\pi \quad (12)$$

$$[\phi(\omega)]_{\Delta H} = -\text{Arctan}\left[\frac{K_1(\omega d)}{K_0(\omega d)} \cdot \frac{Z_0}{H_0}\right] + 2n\pi \quad (13)$$

where $n = 0, \pm 1, \pm 2, \pm 3 \ldots,$

Determination of Parameters

For large values of ωd

$$K_1(\omega d) = K_0(\omega d) = (\pi/2\omega d)^{1/2} \cdot \exp(-\omega d)$$

From equation (11)

$$[A(\omega)]_{\Delta H,\ \text{large}\ \omega d} = 2kVT\,[\omega^{3/2}\,.\,(\pi/2d)^{1/2}]\,.\,\exp\,(\omega d) \tag{14}$$

where $T = [H_0^2 + Z_0^2]^{1/2}$ is the total magnetic field of the earth.

Defining the modified amplitude spectrum

$$[f(\omega)]_{\Delta H} = [A(\omega)]_{\Delta H}\,.\,\omega^{-3/2}$$

$$[f(\omega)]_{\Delta H} = 2kVT\,(\pi/2d)^{1/2}\,.\,\exp\,(-\omega d) \tag{15}$$

A semi-logarithmic plot of $[f(\omega)]_{\Delta H}$ versus ω will be a straight line with slope $-d$. The intercept of the extrapolated linear part on the ω-axis will be

$$\text{Ln } B = \text{Ln } [2kVT\,(\pi/2d)^{1/2}] \tag{16}$$

Similarly from equation (10) it can be shown that:

$$[A(\omega)]_{\Delta Z,\ \text{large}\ \omega d} = 2kV\omega^{3/2}\,.\,(\pi/2d)^{1/2}\,[Z_0^2(1 + 1/\omega d)^2 + H_0^2]^{1/2}\,.\,\exp\,(-\omega d)$$

$$\simeq 2kVT\omega^{3/2}\,.\,(-/2d)^{1/2}\,.\,\exp\,(-\omega d) \tag{17}$$

Equations (14) and (17) appear identical but in reality they differ by a small quantity due to the contribution of the term $(1/\omega d)$.

$$[\phi(\omega)]_{\Delta H,\ \text{large}\ \omega d} = -\,\text{Arctan}\,\frac{Z_0}{H_0} + 2n\pi \tag{18}$$

$$[\phi(\omega)]_{\Delta Z,\ \text{large}\ \omega d} = \text{Arctan}\,\frac{\omega d\,H_0}{(1 + \omega d)Z} + 2n\pi$$

$$\simeq \text{Arctan}\,\frac{H_0}{Z_0} + 2n\pi \tag{19}$$

Equations (18) and (19) show that for large values of ωd or ω the phase spectra for the vertical and horizontal components of the anomalous fields differ by $\pi/2$.

For small values of ωd

$$K_0(\omega d) = \text{Ln}(\omega d)$$

$$K_1(\omega d) = (1/\omega d)$$

$$K_2(\omega d) = (2/\omega^2 d^2)$$

Also, from the recurrence relations of the Bessel functions,

$$K_2(\omega d) = (2/\omega d)\,K_1(\omega d) + K_0(\omega d)$$

Substituting in equation (10),

$$[A(\omega)]_{\Delta Z} = 2kV\omega^2\,[Z_0^2\,\{\,K_2(\omega d) - (1/\omega d)\,K_1(\omega d)\,\}^2 + H_0^2\,\{\,K_1(\omega d)\,\}^2]^{1/2}$$

Therefore, for small values of ωd,

$$[A(\omega)]_{\Delta Z,\ \text{small}\ \omega d} = (2kV/d^2)\,.\,[Z_0^2 + \omega^2 d^2 H_0^2]^{1/2} \tag{20}$$

As ω approaches zero,

$$C = [A(\omega)]_{\Delta Z,\ \omega = 0} = (2kVZ_0/d^2) \tag{21}$$

As d is known, kV can be obtained from (21) if Zo is known. It can be shown from (11) that

$$[A(\omega)]_{\Delta H,\ \omega = 0} = 0 \tag{22}$$

The phase spectra

$$[\phi(\omega)]_{\Delta Z,\ \text{small}\ \omega d} = \text{Arctan}\,(\omega d H_0/Z_0) + 2n\pi \tag{23}$$

$$[\phi(\omega)]_{\Delta Z,\ \omega \to 0} = 0 + 2n\pi$$

and

$$[\phi(\omega)]_{\Delta H,\ \omega \to 0} = -\,\pi/2 + 2n \tag{24}$$

This shows that the phase spectra differ by $\pi/2$ even for small values of ω.

Figures 1 and 2 show the geometry of the system and the components of the anomalous field respectively. Figures 3, 4 and 5 represent the amplitude and phase spectra of the horizontal and vertical components of the anomalous magnetic field due to a sphere, the parameters of which are as follows:

Z_0	=	0.2136×10^5 gammas
H_0	=	0.37×10^5 gammas
k	=	0.06 c.g.s. units
R	=	0.02 Km
d	=	0.1 Km

Resolution of Spectra for Two Spheres

Two identical spheres buried at the same depth (Figure 6a).

The spectrum of the first sphere can be expressed as

$$F_1(\omega) = A_1(\omega)\,.\,\exp[-i\phi_1(\omega)] \tag{25}$$

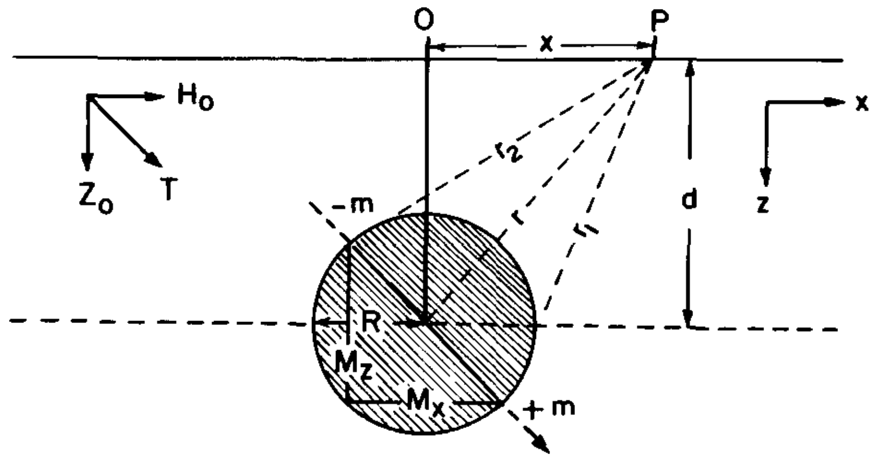

Fig. 1. Geometry of the sphere. 0 = Origin of the co-ordinate system.

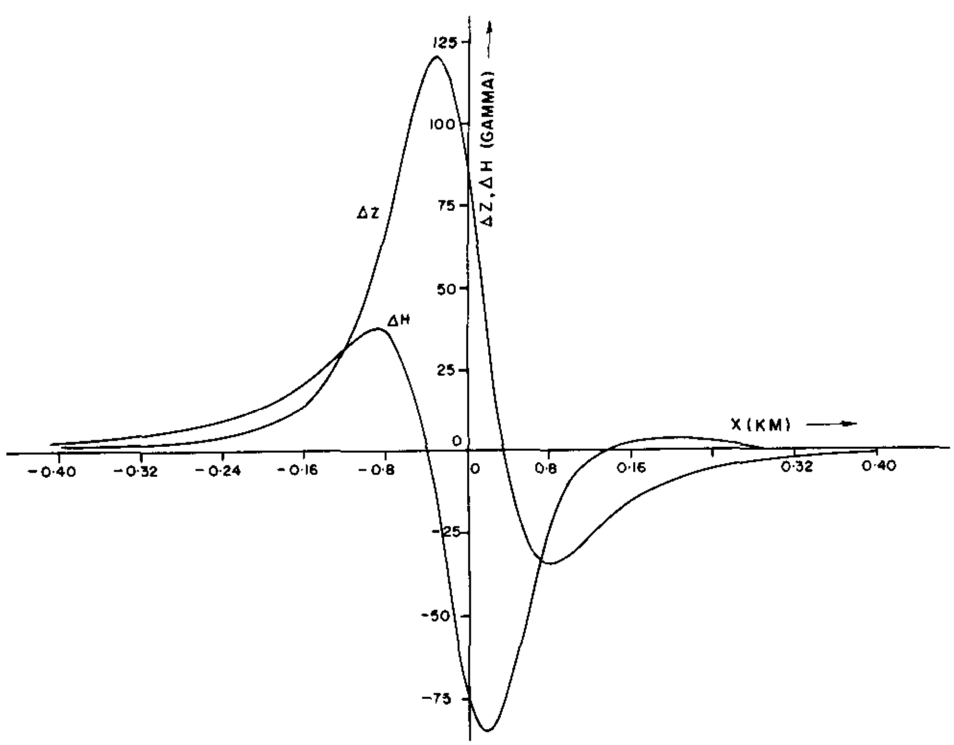

Fig. 2. Horizontal (ΔH) and Vertical (ΔZ) components of the anomalous magnetic field.
k = 0.06 c.g.s. units
d = 0.1 Km
R = 0.02 Km
Z = 0.2136×10^5 gammas
H = 0.37×10^5 gammas.

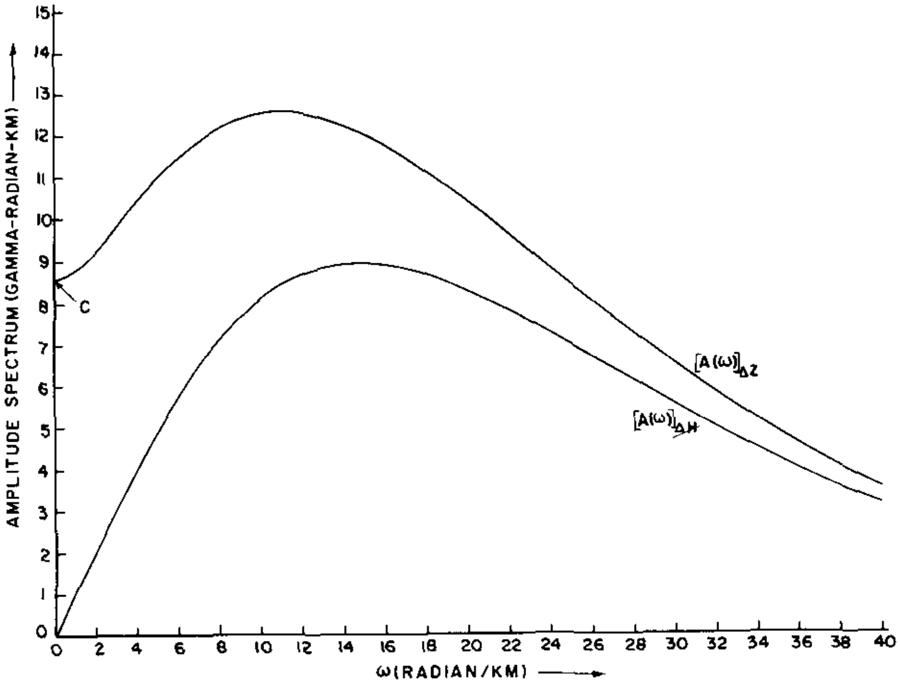

Fig. 3. Amplitude spectra of the vertical and horizontal components of the anomalous magnetic field.
d = 0.1 Km
R = 0.02 Km.

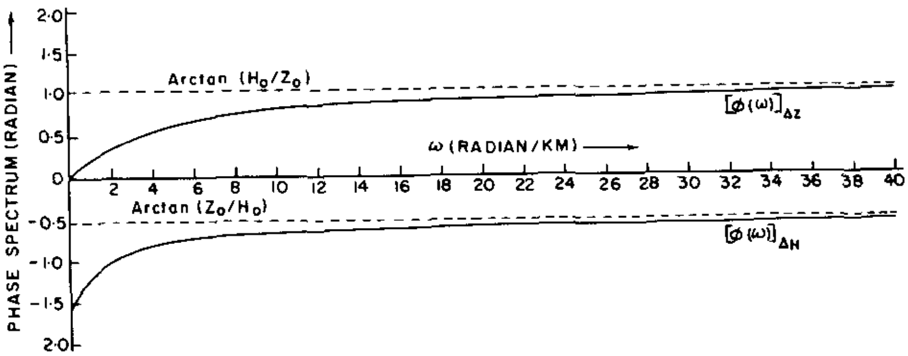

Fig. 4. Phase spectra of the vertical and horizontal components of the anomalous magnetic field.
d = 0.1 Km
R = 0.02 Km.

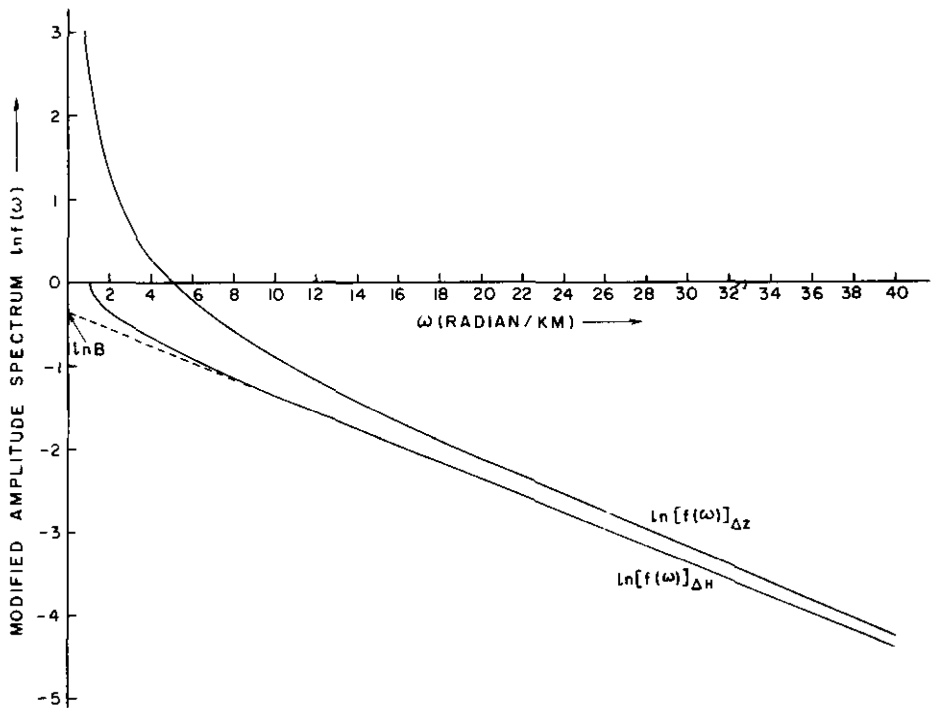

Fig. 5. Modified amplitude spectra.

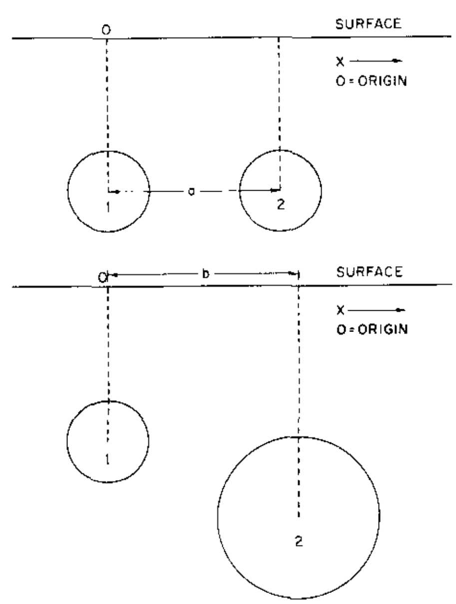

Fig. 6. Geometry of the multiple spheres.
O = origin.

and the spectrum of the second sphere, with reference to the origin, is

$$F_2(\omega) = A_2(\omega) \cdot \exp[-\phi_2(\omega) - i\omega a] \quad (26)$$

As the spheres are at the same depth, and are also of identical size,

$$A_1(\omega) = A_2(\omega) = A(\omega)$$

and

$$\phi_1(\omega) = \phi_2(\omega) = \phi(\omega)$$

From the properties of linear superposition of the Fourier transform, the spectrum for the compound anomaly will be

$$F(\omega) = A(\omega)\,[1 + \exp(-i\omega a)] \cdot \exp[-\phi(\omega)] \quad (27)$$

The amplitude spectrum $|F(\omega)| = [A(\omega)]_{1,2}$ is

$$[A(\omega)]_{1,2} = 2A(\omega)\,\mathrm{Cos}(\omega a/2) \quad (28)$$

From equation (28) it is evident that the amplitude spectrum will be an oscillatory curve, intersecting the ω-axis at $\omega a = (2n+1)\pi$. Also, from the successive values of ω, where $[A(\omega)]_{1,2} = 0$, a can be determined as

$$a = (1/n) \sum_{k=1}^{n} \frac{\pi}{(\omega_{k+1} - \omega_k)} \quad (29)$$

Two spheres of different sizes at different depths (Figure 6b).

In this case, equation (27) can be written in the form

$$F(\omega) = A(\omega) \cdot \exp[-i\phi_1(\omega)] + B(\omega) \cdot \exp[-\omega b - i\phi_2(\omega)]$$

and

$$[A(\omega)]_{1,2} = [\{A(\omega)\}^2 + \{B(\omega)\}^2 + 2A(\omega)B(\omega) \cdot \mathrm{Cos}\{\omega b + \phi_2(\omega) - \phi_1(\omega)\}]^{1/2} \quad (30)$$

The amplitudes of $\mathrm{Cos}\{\omega b + \phi_2(\omega) - \phi_1(\omega)\}$ will oscillate between +1 and −1, and consequently equation (30) will show $[A(\omega) + B(\omega)]$ as peaks and $[A(\omega) - B(\omega)]$ as troughs. From the maxima and minima of $[A(\omega)]_{1,2}$, therefore, the values of $A(\omega)$ and $B(\omega)$ can be resolved.

Conclusions

In the preceding pages the vertical and horizontal components of the anomalous magnetic field generated by a uniformly magnetized sphere are interpreted using Fourier transforms. The sphere is neither an ideal nor a usual representation of a naturally occurring ore body, yet sometimes massive ore bodies are represented by spheres for the purpose of simplified interpretation. It is also shown that only the depth to the centre of the sphere can be determined uniquely, no information regarding the size (V) and the susceptibility contrast (k) is available.

A quantitative estimate of the susceptibility contrast from geological information is necessary to calculate the size of the sphere. The depth to the top of the sphere may be determined by downward continuation of the anomalous field observed on the ground surface. With the depth to the top of the sphere known, the radius may be determined, since the depth to the centre is known.

In analysing the spectra of field data, the origin of the sampled data will be shifted at the flank of the anomaly. This will not affect the amplitude spectrum but will change the phase spectrum by an amount ωa where a is the shift in the origin. It is pertinent to mention that for reasons beyond the authors' control, the theory has not been substantiated by analyses of field data.

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