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REFLECTION AND TRANSMISSION COEFFICIENTS BETWEEN TWO ANELASTIC MEDIA USING ASYMPTOTIC RAY THEORY

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ABSTRACT

The propagation of plane waves in anelastic media has already been examined in depth by Lockett (1962), Cooper and Reiss (1966), Cooper (1967) and Borcherdt (1973, 1977, 1982). In fact, the linear theory of viscoelasticity was used to model the anelasticity of a medium. Their studies show that these plane waves are generally inhomogeneous, i.e., their direction of propagation and maximum attenuation are different. The angle between the propagation vector and the attenuation vector is called the attenuation angle γ. The assumed independence of the boundary conditions on their position along the plane interface leads to Snell's law for viscoelastic media and consists of the conservation of the horizontal components of both the propagation and the attenuation vectors. The physical reality is better approximated, however, by considering a divergent wavefront radiated from a point source. Subsequent application of asymptotic ray theory allows us to obtain Snell's law locally as a consequence of the phase matching applied to the reflected and transmitted waves at the point of incidence. The zeroth order approximation of asymptotic ray theory also leads to different viscoelastic reflection and transmission coefficients. They are, however, fully consistent with those obtained for a plane wave impinging on a plane interface separating two perfectly elastic media (the so-called plane wave reflection/transmission coefficients). These elastic coefficients can be obtained as limit when the quality factor Q tends to infinity. The numerical results computed with both methods, i.e., plane wave and asymptotic ray theory, are presented to demonstrate the differences between the two sets of coefficients.

INTRODUCTION

The reflection/transmission problem of waves propagating in anelastic media is generally modelled using the linear theory of viscoelasticity. Lockett (1962), Cooper and Reiss (1966) and Cooper (1967) examined the reflection and refraction of plane waves at a plane interface between two half spaces of different linear viscoelastic materials. Buchen (1971) investigated the reflection and transmission of *SH*-waves for the same type of solids considering a cylindrical line source. His work assumed slightly dissipative media. A complete analysis of the general case of a plane wave (*SH* and *P-SV*) impinging upon an interface between viscoelastic media was derived by Borcherdt (1977, 1982). Finally Krebes and Hron (1980), Kelamis et al. (1983), Bourbiè and Gonzalez-Serrano (1983), Krebes (1983) and Hearn and Krebes (1990a) actually computed and plotted some viscoelastic reflection and transmission coefficients. In fact, the term "viscoelastic" which has been employed to describe these types of coefficients only means that the linear theory of viscoelasticity was used in their computations. In general, this theory is applied to dissipative media, each being characterised by its *P*- and *S*-wave velocities, its density and its quality factors for *P*- and *S*-waves, which take into account the amplitude decay along the propagation. This loss in amplitude is then modelled by the linear theory of viscoelasticity, even though the media themselves have not been proven to have a purely viscoelastic behaviour. The latter is characterized by the time-dependent Lamé parameters γ and μ in the pertinent stress-strain relation written:

$$\sigma_{ij}(t)=\delta_{ij}\int_{-\infty}^{t}\lambda(t-\tau)\frac{de_{kk}(\tau)}{d\tau}d\tau+2\int_{-\infty}^{t}\mu(t-\tau)\frac{de_{ij}(\tau)}{d\tau}d\tau \quad (1)$$

The above-mentioned viscoelastic coefficient computations were performed by considering a plane wave incident on a plane interface between two viscoelastic media. The results obtained show amplitude and phase differences with the elastic case, mainly in the vicinity of critical angles. In this article, we discuss a different approach to determine reflection and transmission coefficients between two anelastic media. This approach is based on the application of asymptotic ray theory (ART) to the Fourier-transformed basic elastodynamic equation for viscoelastic media (Hron and Nechtschein, submitted). The Fourier image of $\overline{u}(\omega,\vec{r})$ of the displacement vector expanded in the time domain into an asymptotic ray series is actually substituted into the above-mentioned equation. All the calculations are performed in the frequency domain. The final solution in the time domain is obtained after the inverse Fourier transform is carried out. It describes a wave with amplitude decaying exponentially along its raypath and which can be affected by a dispersion relation. Since the asymptotic ray theory is used,

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this approach does not have to consider an incident plane wave and can approximate the case of a nonplanar wavefront incident on a generally curved interface.

The first section briefly describes the traditional plane wave approach used to obtain viscoelastic coefficients. In the second section, the ART approach is presented and the differences between the two methods are examined. Computations of viscoelastic coefficients were performed with both methods for several cases. The results are displayed and analyzed in the last two sections.

Summary of the Plane Wave Approach

The expression for a plane wave in a linearly viscoelastic medium is slightly different from the elastic case because the amplitude decay, due to the viscoelasticity of the medium, must be taken into account. This is achieved by introducing the attenuation vector. A harmonic plane wave propagating in this type of medium is then expressed as

$$Be^{-\vec{A}\cdot\vec{r}}e^{i(\vec{P}\cdot\vec{r}-\omega t)} = e^{i(\vec{k}\cdot\vec{r}-\omega t)} , \qquad (2)$$

where

B is the amplitude at the given frequency;

$\vec{A}$ is the attenuation vector (taking into account the amplitude decay);

$\vec{P}$ is the propagation vector; and

$\vec{k}$ is the complex wave vector ($\vec{k} = \vec{P} + i\vec{A}$).

Figure 1 shows an example of a viscoelastic plane wave. $\vec{P}$ is perpendicular to the planes of constant phase, whereas $\vec{A}$ is perpendicular to the planes of constant amplitude. These two vectors are generally not parallel because the amplitude of a plane wave propagating in a viscoelastic medium can vary along the wavefront (Borcherdt 1977, 1982). In a special case when $\gamma = 0$, i.e., $\vec{P}$ and $\vec{A}$ are parallel, the wave is called a homogeneous plane wave. In the other case when $\gamma \neq 0$, i.e., $\vec{P}$ and $\vec{A}$ are not parallel, which happens to be the most frequent situation, the plane wave is called inhomogeneous.

When a plane wave is incident upon a plane interface (Figure 1), the boundary conditions on this interface are independent of the x position, meaning that the same incidence occurs at any point on the interface. The reflected and transmitted waves are then plane waves also. Figure 2 shows the *SH* case where the subscript $j = 0$ denotes the incident plane wave and subscripts $j = 1, 2$ are used for the pertinent reflected and transmitted waves, respectively. The boundary conditions at the interface $z = 0$, requiring the continuity of the displacement and shear stress across the boundary, are then written as

$$B_0 e^{i(k_{0_x} x)} + B_1 e^{i(k_{1_x} x)} = B_2 e^{i(k_{2_x} x)} \qquad (3)$$

$$M_1 k_{0_z} B_0 e^{i(k_{0_x} x)} - M_1 k_{1_z} B_1 e^{i(k_{1_x} x)} = M_2 k_{2_z} B_2 e^{i(k_{2_x} x)} , \qquad (4)$$

where B_0, B_1 and B_2 are, respectively, the amplitudes of the incident, reflected and transmitted plane waves; k_{0_x}, k_{1_x} and k_{2_x} and k_{0_z}, k_{1_z} *and* k_{2_z} are, respectively, the x and z components of

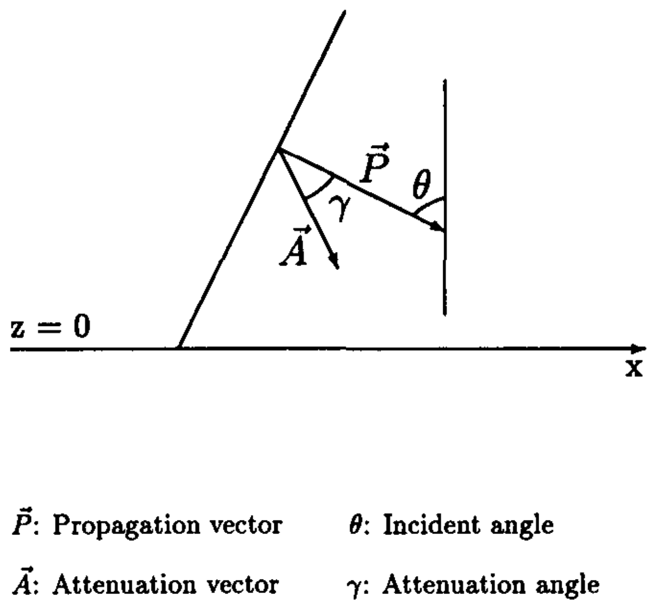

Fig. 1. Example of a viscoelastic plane wave impinging upon an interface.

the corresponding complex wave vectors. M_1 and M_2 are the complex shear moduli of upper and lower media. These complex forms are typical when viscoelastic media are considered. B_0, B_1 and B_2 are independent of x, y and z since the term expressing the amplitude decay is included in the wave vector; consequently for equations (3) and (4) to hold it is required that:

$$k_{0_x} = K_{1_x} = k_{2_x} \Rightarrow P_{0_x} = P_{1_x} = P_{2_x} \text{ and } A_{0_x} = A_{1_x} = A_{2_x} . \qquad (5)$$

This is the so-called Snell's law for viscoelastic media (Borcherdt, 1977, 1982; Wennerberg, 1985). This law clearly has two parts: conservation of the x-component of the propagation vector $\vec{P}$ and of the attenuation vector $\vec{A}$. By knowing the incident and initial attenuation angles of the incident wave, respectively θ_0 and γ_0, the reflection and transmission angles θ_1 and θ_2, the attenuation angle of the reflected wave γ_1 and the attenuation angle of the transmitted wave γ_2 can be determined using this law. For several years, the problem with this method had been the absence of physical criteria to choose the value of γ_0. Therefore, different choices of γ_0 produced different propagation velocities for these types of plane waves and consequently different arrival times and reflection/transmission coefficients (Krebes and Hron, 1980; Krebes, 1983; Hearn and Krebes, 1990a). In 1990, Hearn and Krebes suggested that Fermat's principle could be used to determine the proper choice of γ_0. A unique set of reflection/transmission coefficients is then obtained at the price of the inhomogeneous plane waves and complex angles which inevitably leads to the concept of complex rays whose physical interpretation is not always elementary (see Hearn and Krebes, 1990a, b).

ART Approach

Plane waves have, of course, plane wavefronts. The reflection/transmission problem is probably better approximated by considering a nonplanar wavefront leading to different

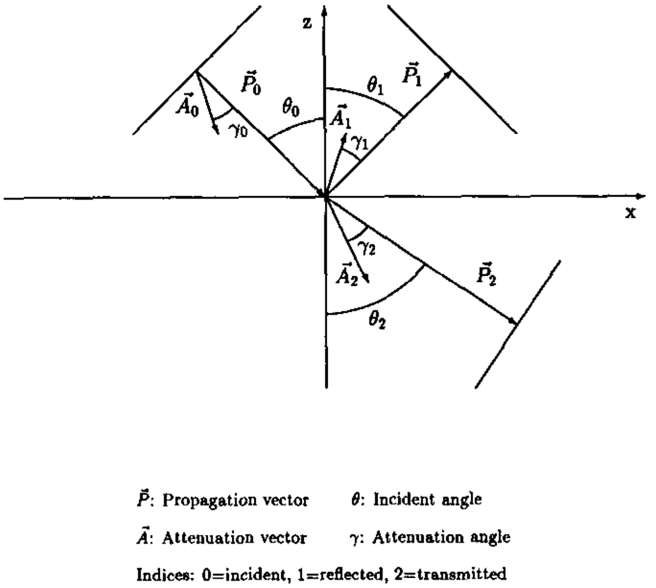

Fig. 2. Incident, reflected and transmitted plane waves at a boundary between two viscoelastic media (*SH* case).

geometries and ranges of validity of boundary conditions (Figure 3). In particular, the application of the so-called "phase-matching" which is a standard procedure in asymptotic ray theory (ART) leads to the requirement that the mathematical relations expressing the boundary conditions be valid only in the immediate neighbourhood of the point of incidence. ART will now be used as a basis for the computations of viscoelastic reflection and transmission coefficients. All the following derivations have been made with the ART zeroth order terms.

When a ray is traced, ART only considers the energy travelling in the vicinity of the central ray of the ray tube (Figure 4). Consequently, at the point of incidence on the interface, ART requires the continuity of the displacement and the stress only in the immediate vicinity of the point of incidence. There is, therefore, no need to keep the relations expressing the boundary conditions the same along the entire interface, which means that any shape of boundary can be considered. Since we only deal with the energy travelling along the central ray of the ray tube, the amplitude decay has to be expressed along the ray. For a ray travelling in a linear viscoelastic medium and for a particular frequency f, this is given by (Aki and Richards, 1980):

$$e^{-\frac{\omega}{2cQ}(s-s_0)}, \tag{6}$$

where

c is the phase velocity at the frequency f;

Q is the quality factor of the medium at the frequency f;

$(s - s_0)$ is the length of the ray segment in the medium; and

$\omega = 2\pi f$ is the so-called angular frequency of the harmonic source.

This decay term depends on the frequency and therefore has to be calculated for each frequency present in the source frequency spectrum. The principle of the method is shown in Figure 5 and can be described in the following way: from an incident amplitude $V_{0_{s_0}}$ given at a particular frequency, the actual amplitude reaching the point of incidence at the interface V_0 is calculated as:

$$\frac{V_{0_{s_0}}}{L} e^{-\frac{\omega}{2cQ}(s-s_0)} = V_0, \tag{7}$$

where L is the geometrical spreading between the source and the point of incidence. Then using the zeroth order of ART, the boundary conditions can be obtained. For example, in the *SH* case:

continuity of displacement:

$$V_0 + V_1 = V_2 \tag{8}$$

continuity of stress:

$$M_1\left[\frac{\cos\theta_0}{v_0}V_0\right] - M_1\left[\frac{\cos\theta_1}{v_1}V_1\right] = M_2\left[\frac{\cos\theta_2}{v_2}V_2\right], \tag{9}$$

where M_1 and M_2 are the complex shear moduli of upper and lower media at a particular frequency; θ_0, θ_1 and θ_2 are the real incident, reflection and transmission angles and v_0, which is equal to v_1, and v_2 are the real phase velocities of the upper and lower media. The complex shear modulus of each medium is obtained from

$$M_j(\omega) = \rho_j v_{c_j}^2(\omega), \tag{10}$$

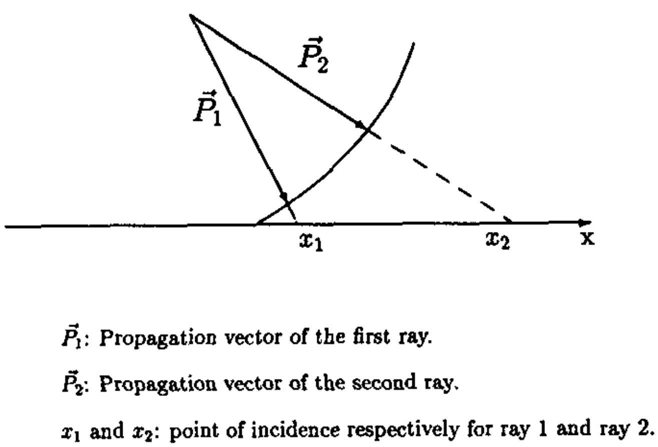

Fig. 3. Example of a nonplanar wavefront with different angles of incidence.

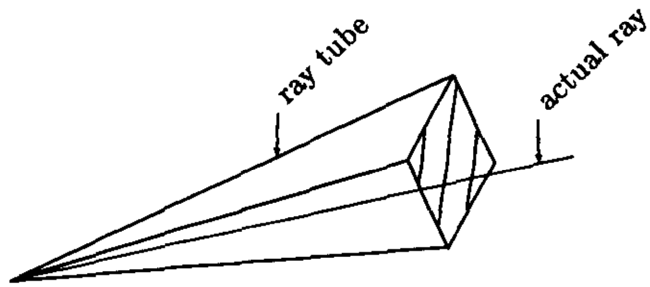

Fig. 4. Concept of a ray tube used in asymptotic ray theory (ART).

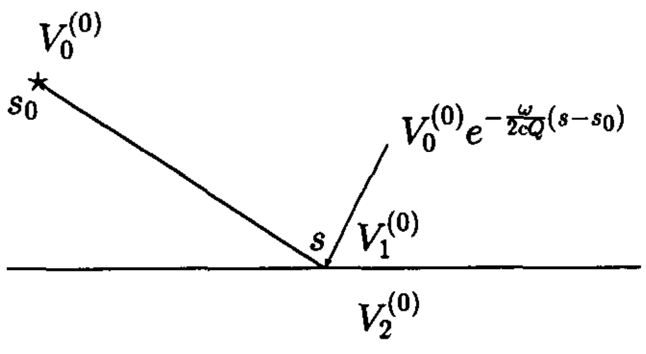

Fig. 5. Principle of the asymptotic ray theory approach. The zeroth order of the actual amplitude reaching the point of incidence is $V_0^{(0)} e^{-\frac{\omega}{2cQ}(s-s_0)}$, whereas $V_1^{(0)}$ and $V_2^{(0)}$ are the zeroth-order terms representing the reflected and transmitted amplitudes. Please note that no geometrical spreading is assumed in this case.

where ρ_j and v_{c_j} are, respectively, the density and the complex velocity of the jth medium, v_{c_j}, being defined by

$$\frac{1}{v_{c_j}(\omega)} = \frac{1}{v_j(\omega)} + \frac{i}{2v_j(\omega)Q_j(\omega)} \tag{11}$$

(Hron and Nechtschein, submitted). Equations (8) and (9) have to be valid for each frequency contained in the source frequency spectrum. In these two equations, V_j, $j = 0, 1, 2$ are the ray amplitudes of *SH* waves at the same points of incidence even though V_2 is taken on the opposite side of the boundary. Hence, these amplitudes have not been subjected to any decay caused by the propagation and they are affected by the viscoelasticity of the media because of the presence of the complex-valued shear moduli. From equations (8) and (9) the reflection and transmission are easily determined. These two equations are completely independent of the attenuation angle γ, and since the usual real value phase velocity is used, Fermat's principle is always satisfied and the original Snell's law is obtained from the process of phase matching. The frequency dependence of equations (8) and (9) comes from the fact that ART is, in fact, applied to the Fourier transform of the elastodynamic equation for linear viscoelastic media, as mentioned in the introduction (Hron and Nechtschein, submitted). With this approach, all calculations, including those for coefficients, are done in the frequency domain and, consequently, are frequency dependent. The physically meaningful solution for the displacement vector carried by the wave propagating in linear viscoelastic media is then obtained after the following inverse Fourier transform

$$\vec{v}_j(t,\vec{r}) = \frac{1}{\pi}\mathrm{Re}\int_{\omega_0}^{\infty} S(\omega)\sum_{n=0}^{\infty}\vec{V}_j^{(n)}\frac{e^{-i\omega(t-\tau(\vec{r},\omega))}}{(-i\omega)^n}d\omega \quad j = 0,1,2\,, \tag{12}$$

is performed (Hron and Nechtschein, submitted). Here $S(\omega)$ is the source pulse frequency spectrum and $\vec{V}_j^0$ is the zero amplitude term of the ray series considered in boundary conditions (8) and (9). ω_0 is positive and is the lower frequency bound of integration needed to avoid any difficulties with the convergence for the integral. It is easy to see that the asymptotic series in (12) can be regarded as a ray series due to a monochromatic harmonic wave with the source function $e^{-i\omega t}$ carrying a transformed displacement vector

$$\vec{v}'(\omega,\vec{r}) = \sum_{n=0}^{\infty}\vec{V}^{(n)}(\omega,\vec{r})\frac{e^{-i\omega(t-\tau(\vec{r},\omega))}}{(-i\omega)^n}. \tag{13}$$

The integration of monochromatic harmonic waves over frequency, described in (12), implies that the amplitude term of the ray series (13) is now frequency dependent. Since the reflection and transmission coefficients are of primary importance in the determination of this amplitude term, they have to be computed at each frequency present in the source frequency spectrum. The change in the actual values of the coefficients as a function of frequency is related to the dispersion relation taken into consideration when characterizing the linear viscoelastic media.

With this method of calculating the reflection and transmission coefficients, a smooth transition between the numerical values pertinent to the perfectly elastic and viscoelastic cases is obtained. The main differences between the two methods are summarized in Table 1. The plane wave approach outlined in this table was used to compute the plane wave *P-SV* coefficients for the next section following the method of Hearn and Krebes (1990).

Numerical Results

In this section the *P-SV* reflection and transmission coefficients computed with both methods are now compared. The *SH* case was used above only to show the principle of both approaches but the *P-SV* case was chosen to display coefficients because more coefficients can be compared. The system to be solved to obtain the *P-SV* coefficients depends on the mode (*P* or *SV*) of wave propagation carried by the incident wave. For an incident *P*-wave, the system describing the boundary conditions is:

$$\sin\theta_0 = -P1P1\sin\theta_1 - P1S1\cos\theta_3 + \sin\theta_2 P1P2 + \cos\theta_4 P1S2$$

Table 1. *Main differences between the plane wave and ART* approaches to calculate viscoelastic reflection and transmission coefficients.

Plane Wave Approach	ART Approach
Lamè parameters Λ and M, the velocities and the angles are all complex.	Lamè parameters Λ and M are complex but velocities and angles are real.
The term taking into account the amplitude decay is included in the wave vector and therefore present in the boundary conditions. ⇓	The term taking into account the amplitude decay is only used to calculate the actual amplitude incident on the interface. ⇓
Special Snell's law with the attenuation angle γ which influences the direction of $\vec{P}$ and the velocities of reflected and transmitted rays.	Original Snell's law, i.e., only phase matching, is used. The direction of $\vec{P}$, the velocities of reflected and transmitted rays, are not affected by Q. The kinematic is conserved.

$$\cos\theta_0 = P1P1\cos\theta_1 - P1S1\sin\theta_3 + \cos\theta_2 P1P2 - \sin\theta_4 P1S2$$

$$M_0\frac{\sin 2\theta_0}{v_0} = M_1\frac{\sin 2\theta_1}{v_1}P1P1 + M_3\frac{\cos 2\theta_3}{v_3}P1S1$$

$$+M_2\frac{\sin 2\theta_2}{v_2}P1P2 + M_4\frac{\cos 2\theta_4}{v_4}P1S2$$

$$\left(\frac{\Lambda_0 + 2M_0\cos^2\theta_0}{v_0}\right) = -P1P1\left(\frac{\Lambda_1 + 2M_1\cos^2\theta_1}{v_1}\right) + P1S1M_3\frac{\sin 2\theta_3}{v_3}$$

$$+P1P2\left(\frac{\Lambda_2 + 2M_2\cos^2\theta_2}{v_2}\right) - P1S2M_4\frac{\sin 2\theta_4}{v_4}.$$

In the case of an incident S-wave, a similar system describes the boundary conditions:

$$\cos\theta_0 = -S1P1\sin\theta_1 - S1S1\cos\theta_3 + \sin\theta_2 S1P2 + \cos\theta_4 S1S2$$

$$\sin\theta_0 = -S1P1\cos\theta_1 + S1S1\sin\theta_3 - \cos\theta_2 S1P2 + \sin\theta_4 S1S2$$

$$M_0\frac{\cos 2\theta_0}{v_0} = M_1\frac{\sin 2\theta_1}{v_1}S1P1 + M_3\frac{\cos 2\theta_3}{v_3}S1S1$$

$$+M_2\frac{\sin 2\theta_2}{v_2}S1P2 + M_4\frac{\cos 2\theta_4}{v_4}S1S2$$

$$\left(\frac{M_0\sin 2\theta_0}{v_0}\right) = S1P1\left(\frac{\Lambda_1 + 2M_1\cos^2\theta_1}{v_1}\right) - S1S1M_3\frac{\sin 2\theta_3}{v_3}$$

$$-S1P2\left(\frac{\Lambda_2 + 2M_2\cos^2\theta_2}{v_2}\right) + S1S2M_4\frac{\sin 2\theta_4}{v_4},$$

where

θ_0: angle of incidence;
θ_1: angle of reflection for P-wave;
θ_2: angle of transmission for P-wave;
θ_3: angle of reflection for S-wave;
θ_4: angle of transmission for S-wave;
v_0: phase velocity of the incident wave;
v_1: phase velocity of the reflected P-wave;
v_2: phase velocity of the transmitted P-wave;
v_3: phase velocity of the reflected S-wave;
v_4: phase velocity of the transmitted S-wave;
M_0=M_1=M_3 and Λ_0=Λ_1 stand for complex Lamè parameters in the upper medium; and
M_2=M_4 and Λ_2 denote the complex Lamè parameters in the lower medium.

Tables 2 and 3 present models selected to compute viscoelastic coefficients. For a single boundary there are 16 coefficients all together, 8 for the incidence from above and 8 for the incidence from below. In this paper, only some of them were selected for our discussion due to space limitation. The results displayed for $P1P1$, $P1S1$ and $P1P2$ coefficients were computed with the model described in Table 2, whereas the ones displayed for $S1S1$ were obtained with another model (Table 3). For the model in Table 2, several computations were performed by keeping the same velocities and densities but by changing the quality factors. The purpose of this exercise was to check the "continuity of coefficients" expected when the dissipative properties of both half-spaces are decreased thereby approaching the ideal elastic case. In such a process, it is reasonable to expect the amplitude and the phase curves of the viscoelastic coefficients to get closer and closer to those obtained for the elastic case. Both methods were used and the test results are shown here for the $P1P1$, $P1S1$ and $P1P2$ coefficients (Figures 6, 7, 8). The 'e' curve represents the elastic case, the #1 and #2 curves were respectively obtained for the models 1 and 2 mentioned in Table 2. Model 1 is less attenuating than model 2 because the quality factors of model 1 are higher. For the reflection coefficients $P1P1$ and $P1S1$, the results obtained with the plane wave approach show that the amplitude curves are smooth, being continuous over the entire range. For $P1S1$, these curves are similar for all the models. The same comment is true for $P1P1$ except the vicinity of the critical angle. In this part of the graph (Figure 6), the three curves are distinct but the differences between the elastic and the dissipative models decrease when the media is less attenuating (#1 curve is closer to the 'e' curve than the #2 curve). Unfortunately, the phase curves do not show this continuity. There exists a phase shift between the 'e' curve and the #1 and #2 curves beyond the critical angle. Additional computations showed that the phase curve obtained for the absorbing media does not continuously approach the elastic one when the quality factors are increased (Table 4) but, instead, suddenly changes to become similar to the elastic phase curve, at particular Q values (Figure 9). On the other hand, with the ART approach, continuity exists for both amplitude and phase. For each reflection coefficient the viscoelastic amplitude and phase curves become closer to the elastic ones when the media on both sides of the interface are less attenuating. The actual differences between the three curves, for the amplitude and for the phase, are very small and cannot even be seen on the graphs (Figures 6, 7). For $P1P1$, the loss in amplitude around the critical angle is not present anymore. There are some slight differences between viscoelastic and elastic phase curves for each coefficient. This is due to the introduction of complex moduli in the equations describing the boundary conditions. The undisturbed continuity of both amplitude and phase curves is clearly evident in the graphs, thereby confirming our physical intuition.

Figure 8 shows the results obtained for $P1P2$. For each method, the viscoelastic amplitude and phase curves are extremely close to the elastic ones. Therefore, both approaches seem to give similar results, with no continuity problem for the plane wave approach in the transmission case.

Unfortunately, another type of problem can occur for the

Table 2. Set of 3 models used to perform the continuity test in which the amplitude and phase curves of the viscoelastic coefficients should move closer and closer to the amplitude and phase curves of the elastic coefficients as the two media across the interface become less and less attenuating. The top model is the elastic case: no attenuation, all the quality factors are set to infinity. Case 1 (middle) is an intermediate attenuating case, the media on both sides of the interface are attenuating but not as much as the next case. Case 2 (bottom) is the most attenuating case with the quality factors being the lowest in all three models. For all these cases, the *S*- and *P*-wave velocities and the density of each medium are kept the same.

Elastic Case	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	1.9	1.0	1.0	∞	∞
Layer 2	2.1	1.2	1.2	∞	∞

Case 1	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	1.9	1.0	1.0	81	50
Layer 2	2.1	1.2	1.2	92	70

Case 2	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	1.9	1.0	1.0	60	30
Layer 2	2.1	1.0	1.0	80	40

Table 3. Example of model which produces unusual behaviour of the modulus of the viscoelastic *S1S1* coefficient displayed versus the angle of incidence.

S1S1 Model	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	4.2	2.4	2.1	200	100
Layer 2	6.4	3.5	2.6	300	200

Table 4. Model used to show a sudden switch of the viscoelastic phase curve obtained with the plane wave approach: case 1 – case producing a phase difference between viscoelastic and elastic phase curves; case 2 – case which does not produce the phase difference mentioned in case 1. The quality factors have been slightly modified from those of case 1.

Case 1	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	1.9	1.0	1.0	1000	990
Layer 2	2.1	1.2	1.2	1001	995

Case 2	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	1.9	1.0	1.0	1000	990
Layer 2	2.1	1.0	1.0	1000	995

computation of viscoelastic reflection coefficients when the plane wave method is used. In our experience with many different anelastic models, the viscoelastic *S1S1* coefficient amplitude curve computed with the plane wave approach sometimes exhibits unusual behaviour around the *S1S2* critical incidence. An example of these unusual amplitude curves obtained for the *S1S1* coefficient is displayed in Figure 10. The viscoelastic amplitude curve is labelled curve '*a*' and was obtained with the model described in Table 3. Curve '*a*' goes up and down around the *S1S2* critical incidence before steadily increasing to 1. The rest of the curve is very similar to the elastic case represented here by curve '*e*'. A phase difference again exists between the viscoelastic and elastic phase curves. All these amplitude irregularities do not appear when the ART approach is used. For both amplitude and phase, the viscoelastic and elastic curves are extremely close to each other.

Example of an Unacceptable Amplitude Growth for Transmitted Plane Waves

For a viscoelastic plane wave, the ray parameter p is a complex number and the Cartesian coordinates of the propagation vector $\vec{P}$ and the attenuation vector $\vec{A}$ are given by:

$$\vec{P} = \omega(\operatorname{Re} p, 0, -\operatorname{Re}\xi), \quad \vec{A} = \omega(\operatorname{Im} p, 0, -\operatorname{Im}\xi), \tag{14}$$

where

$$\xi = \sqrt{\frac{1}{c^2} - p^2}\ ,$$

with p being the ray parameter and c the complex wave speed (Krebes, 1983; Richards, 1984; Wennerberg, 1985). For the reflection/transmission problem, the signs of some $\vec{P}$ components are imposed because of the plane wave propagation directions (Figure 11). For the z components of $\vec{P}$, the root which has the positive real part is always required. This means we have no control on the sign of Imξ. Depending on the values chosen for input parameters to calculate coefficients, the following situation can happen: the amplitude increases when the plane wave moves away from the interface. Richards (1984) detailed such a case with an *S*-wave incident from the lower medium (Figure 12). Using the model given by Richards (1984) (Table 5), the corresponding transmitted *P*-wave has an amplitude growth away from the interface when the take-off angle is greater than 13.3°. For these precritical incidence angles, the amplitude at *Y'* is higher than that at *X'* (see Richards, 1984). This situation is not acceptable since the amplitude is not supposed to grow indefinitely with the distance from the interface. Richards (1984) suggested that the correct value of the transmission coefficient in such a case of amplitude growth can be obtained if the transmitted wave is evaluated at large $|z|$. He also mentioned that in order to examine the problem in general, some allowance for curved wavefronts is required (1984). This is what is being done using ART since a nonplanar wavefront striking the interface is considered (Figure 12). The transmission coefficient is then directly determined by solving the classical system of four equations. Figure 13 shows the results obtained by both methods. They are similar for precritical incidence. For the postcritical incidence, i.e., $\theta > 28.1°$, ART does not produce a coefficient because the geometrical ray does not exist anymore. It is also worth mentioning that Richards' results were computed with a 0° initial attenuation angle γ and not with the initial attenuation angle which satisfies Fermat's principle.

Conclusions

Reflection and transmission coefficients between two anelastic media were computed with the plane wave approach and with asymptotic ray theory (ART). For elastic media, the two methods give identical results because the attenuation vector of a plane wave is always equal to 0, so that the same Snell's law is used. This is not the case when the plane interface separates anelastic media. Then the attenuation vector of

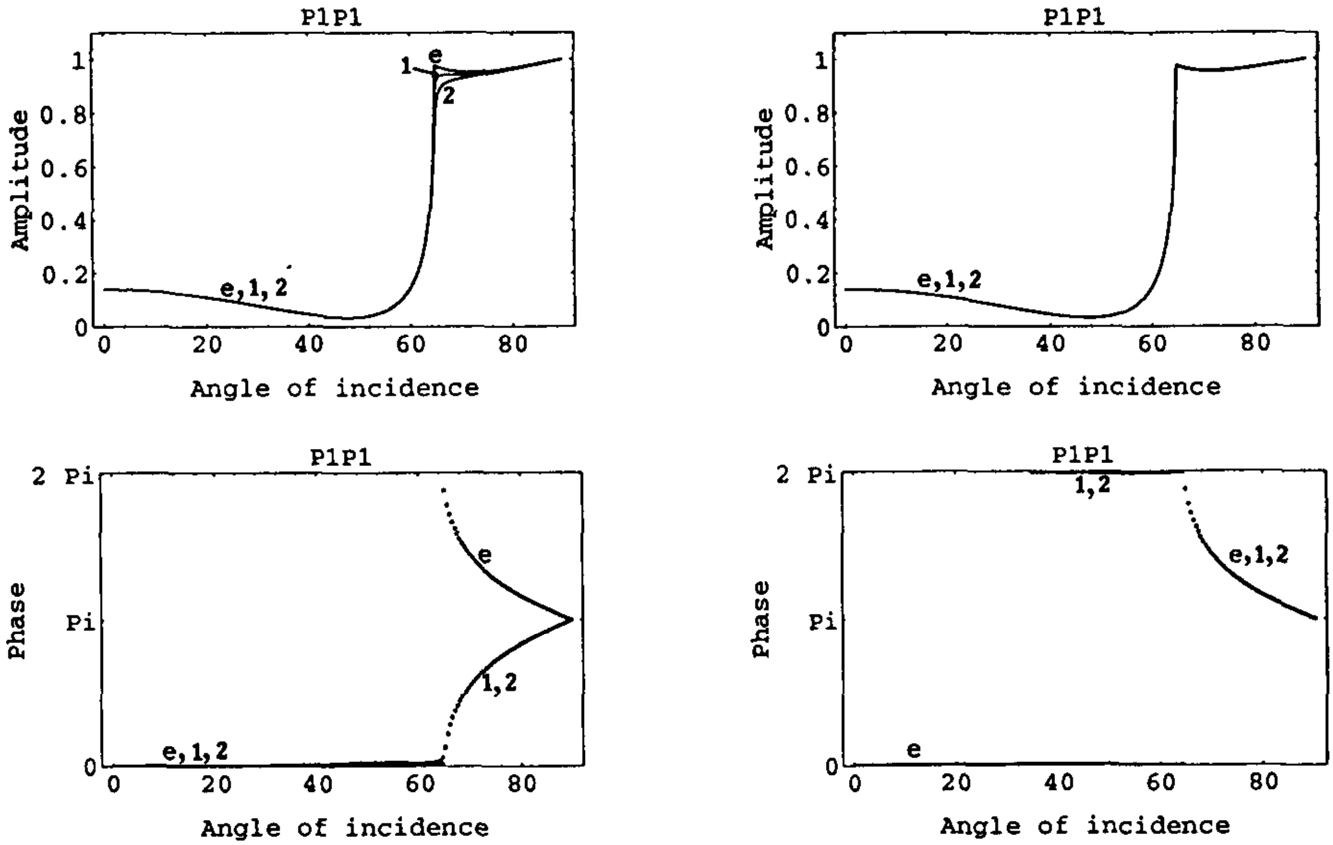

Fig. 6. Amplitude and phase curves for the *P1P1* coefficient. The results obtained with the plane wave approach are on the left and those obtained with ART are on the right. The models described in Table 2 were used. The '*e*' curve represents the elastic case and the #1 and #2 curves were respectively computed for case 1 and case 2 from Table 2.

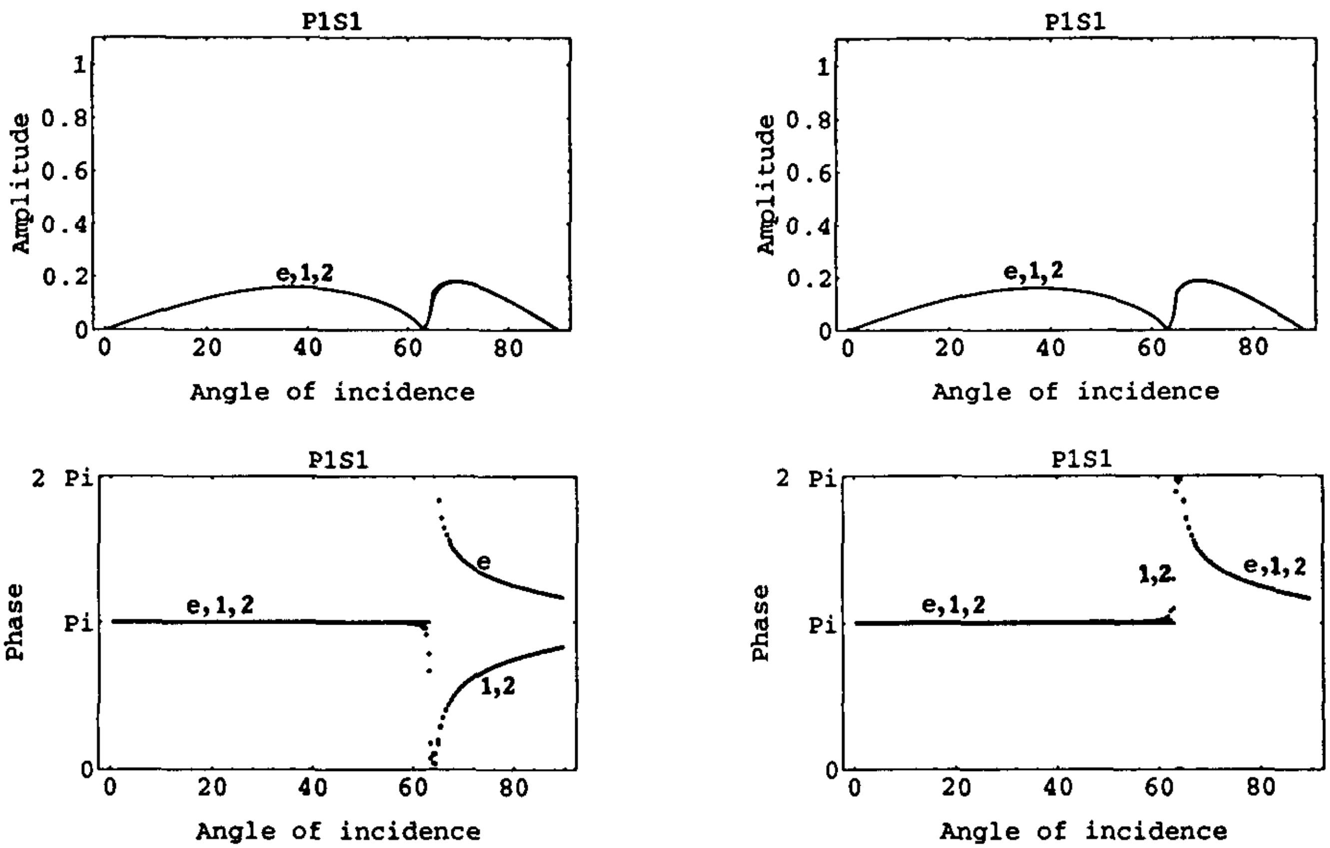

Fig. 7. Amplitude and phase curves for the *P1S1* coefficient. The results obtained with the plane wave approach are on the left and those obtained with ART are on the right. The same models and the same curve notations as in Figure 6 are used.

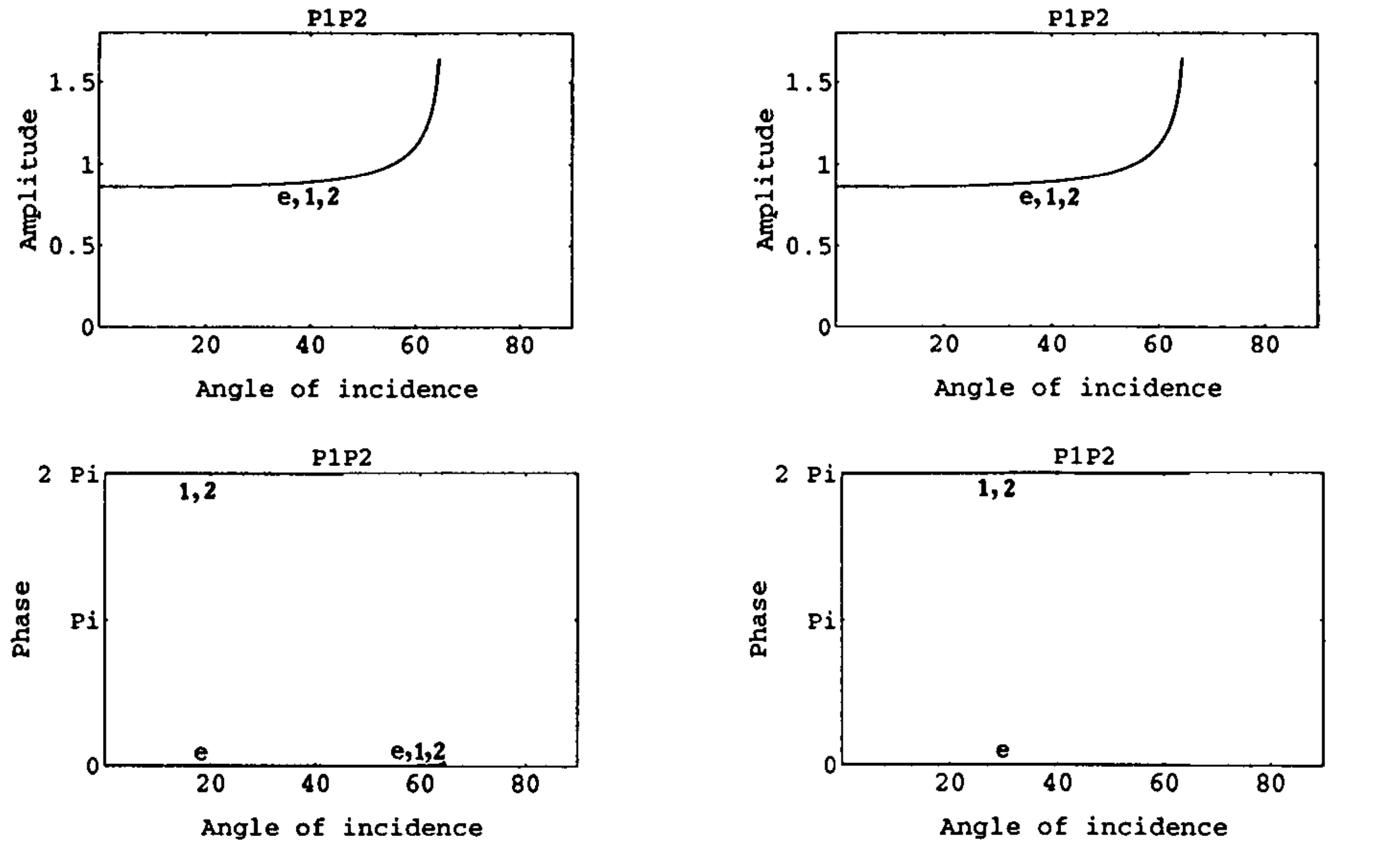

Fig. 8. Amplitude and phase curves for the *P1P2* coefficient. The results obtained with the plane wave approach are on the left and those obtained with ART are on the right. The same models and the same curve notations as in Figure 6 are used.

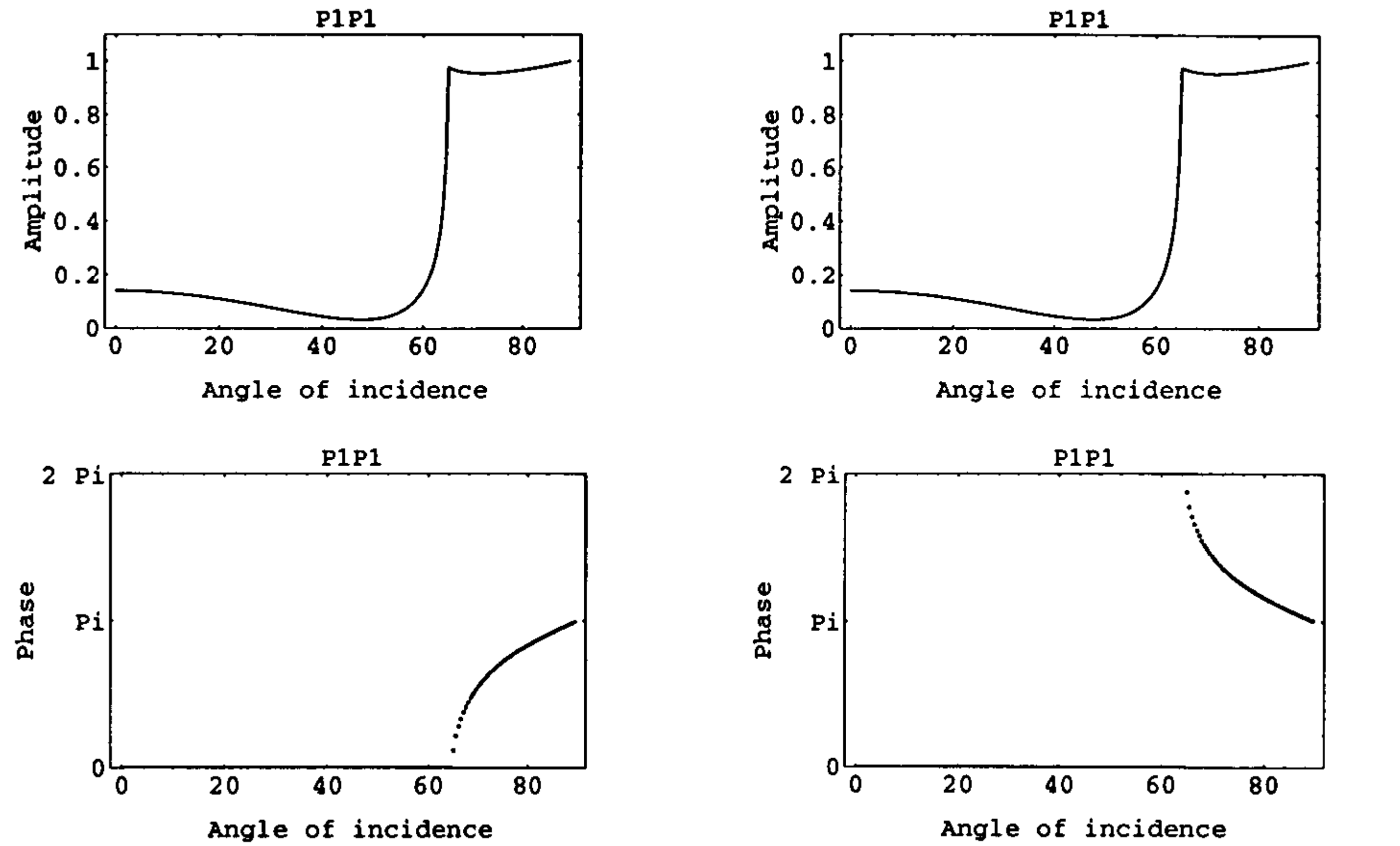

Fig. 9. Example showing a sudden phase shift of the viscoelastic phase curve for a very slight change in the quality factors. This shift is obtained with the plane wave method only. The models described in Table 4 were used; case 1 is on the left and case 2 is on the right.

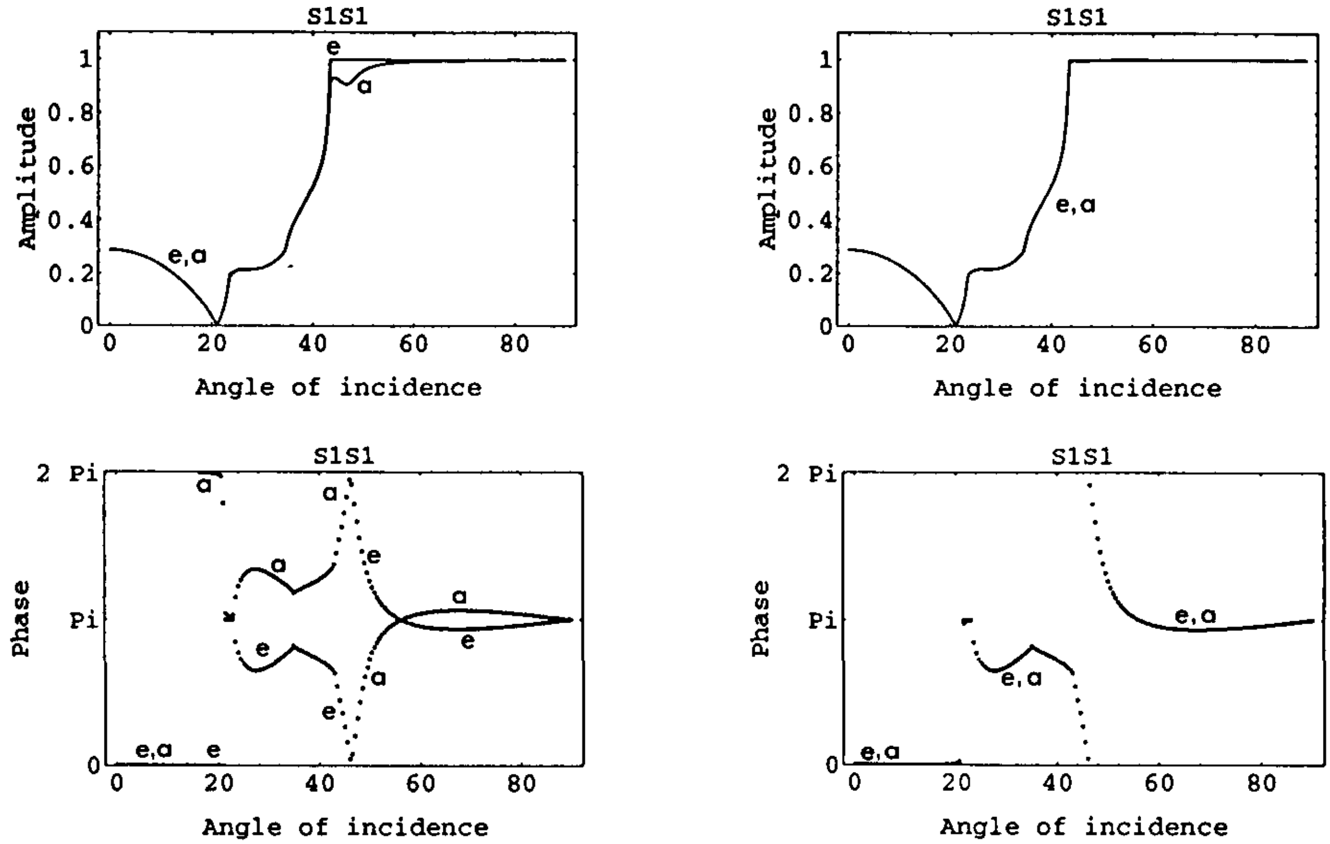

Fig. 10. Amplitude and phase curves for the $S1S1$ coefficient. The results obtained with the plane wave approach are on the left and those obtained with ART are on the right. The models described in Table 3 were used. The '*e*' curve represents the elastic case and the '*a*' curve was computed for the viscoelastic case.

a plane wave is different from 0 and the plane wave approach requires a special form of Snell's law guaranteeing the continuities of x components of both $\vec{P}$ and $\vec{A}$ vectors across the interface. In the ART approach, the form of Snell's law remains the same as the original. The numerical values of the coefficients obtained with the help of ART also seem to be more physical. We see that when the two media on both sides of the interface become less and less dissipative, the amplitude and phase curves of the coefficients computed with ART become closer and closer to the curves obtained for the case when both media are perfectly elastic. With the plane wave approach, such continuity only seems to exist for the amplitude. The phase curve experiences some phase shifts for particular quality factor values. Furthermore, some unusual shapes are observed around critical incidence on the amplitude curve of the $S1S1$ coefficient when it is computed for an incident plane wave. These curve shapes do not exist when this same coefficient is calculated with ART, with the amplitude curve staying very close to the elastic amplitude curve. Finally, for some special cases of a physically unacceptable amplitude growth, the plane wave approach requires extra calculations (Richards, 1984) away from the boundary to evaluate the proper value of transmission coefficients. This is never the case when ART is used, since the transmission coefficients are always determined properly at the interface and the situation of amplitude growth away from the boundary can not take place. For a critically reflected/transmitted ray, the corresponding propagation angle is complex, resulting in an exponential decay along the z direction, since the same Snell's law as for the elastic case is used. This leads to the presence of an evanescent wave which is in fact an inhomogeneous wave with a 90° attenuation angle. Finally, we can also consider "locally plane wavefronts" in our treatment without any additional difficulty, using our system of reflection and transmission coefficients. This is due to the fact that in our method we have never considered plane wavefronts extending to infinity, which is the case of the plane wave approach.

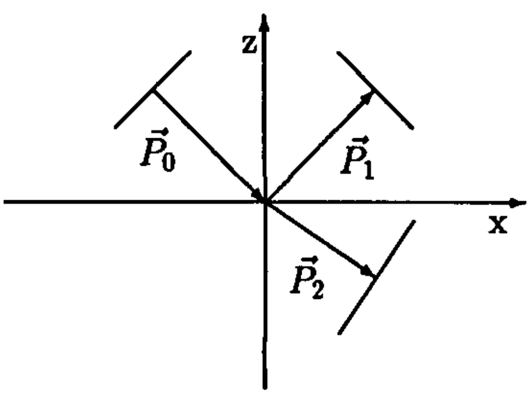

Fig. 11. Example of incident, reflected and transmitted plane waves (*SH* case) to show that the radiation conditions impose the Cartesian coordinate signs of $\vec{P}_0$, $\vec{P}_1$ and $\vec{P}_2$ standing for the propagation vectors of the incident, reflected and transmitted plane waves, respectively.

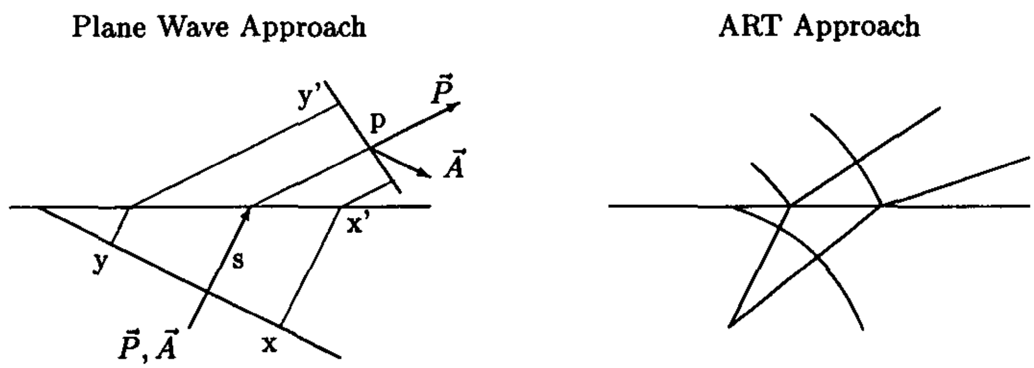

Fig. 12. Illustration showing the geometrical differences between the plane wave approach (left) and the ART approach (right) for the amplitude growth case described by Richards (1984). This amplitude growth problem only occurs with the plane wave method.

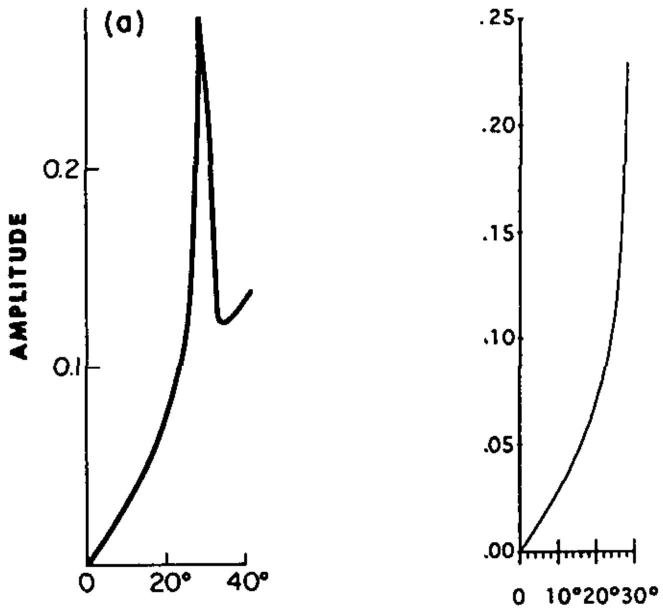

Fig. 13. Amplitude curves for the viscoelastic *S2P1* coefficient used by Richards (1984) to investigate an amplitude growth case occurring with the plane wave theory. The amplitude curve obtained by Richards is on the left and that obtained with ART is on the right. They were computed for the model described in Table 5.

Table 5. Model used by Richards (1984) to obtain a case of amplitude growth for transmitted plane wave.

Richards' Model	V_P(km/s)	V_S(km/s)	ρ(g/cc)	Q_P	Q_S
Layer 1	9.7	5.3	1.084	630	400
Layer 2	8.55	4.58	1.0	250	150

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