MINIMUM-PHASE*

By

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ABSTRACT

The concept of minimum-phase is reviewed from basic considerations. The question of the physical realizability of minimum and non-minimum-phase systems, and the condition required for the determination of the phase response of a system from the amplitude response, are discussed. It is shown that physical realizability does not imply minimum-phase.

INTRODUCTION

The term, minimum-phase, appears with increasing regularity in geophysical literature. It is a concept which has been borrowed from the field of electrical network analysis, where it was first introduced by Bode (1945). We have found that geophysicists are generally less familiar with this concept than are electrical engineers. Such questions as: "Are non-minimum-phase systems physically realizable?" or "Under what conditions may the phase response of a filter be deduced from its amplitude response?", are often asked.

We feel it is therefore appropriate to review the concept of minimum-phase from the frequency and time domain points of view, and to illustrate our discussion by considering an elementary filtering circuit.

THEORY

In the discussion to follow we will limit ourselves to a consideration of systems composed of lumped, passive components; in other words, we consider systems with inductance, capacitance and resistance only.

Pole-Zero Representation of a Transfer Function

The transfer function of a system, T (s) is defined as the transform of the output divided by the transform of the input, and may be represented as the ratio of two polynomials. In general s is a complex number, $\sigma + i\omega$. In the particular case where the Fourier transform is used, $\sigma = o$ and T(s) becomes $T(i\omega)$, where ω is the angular frequency.

$T(i\omega)$ may be composed of real and imaginary parts:

$$T(i\omega) = G(\omega) + iB(\omega) = A(\omega)e^{i\phi(\omega)}$$

where $A(\omega) = \sqrt{G^2(\omega) + B^2(\omega)}$ is the amplitude response of the system

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and $\phi(\omega) = \tan^{-1}\dfrac{B(\omega)}{G(\omega)}$ is the phase response of the system.

The representation of a transfer function in the complex S plane is a very useful one and we will use it in our discussion.

As an illus‘‘‘tion of this representation, let us consider the low-pass filter (Fig. 1) for which $T(s) = \dfrac{1}{1 + s\tau}$ where $\tau = CR$ is the time con-

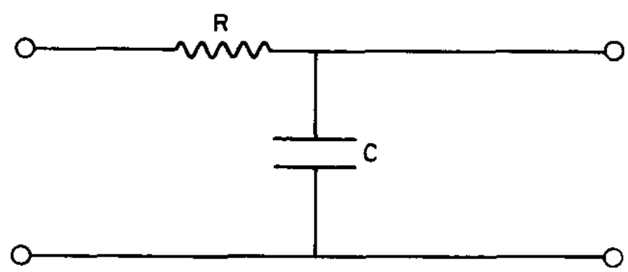

FIG. 1—A low-pass filter.

stant in seconds. Since T(s) becomes infinite at $s = -\dfrac{1}{\tau}$, we say that T(s) has a pole at this value of s. Further, since the numerator of T(s) can never be zero, the transfer function of a low-pass filter does not possess zeros in the S plane.

The S plane representation for this low-pass filter (Fig. 2) depicts the pole-zero configuration of the transfer function and may be used to determine the behaviour of $A(\omega)$ and $\phi(\omega)$.

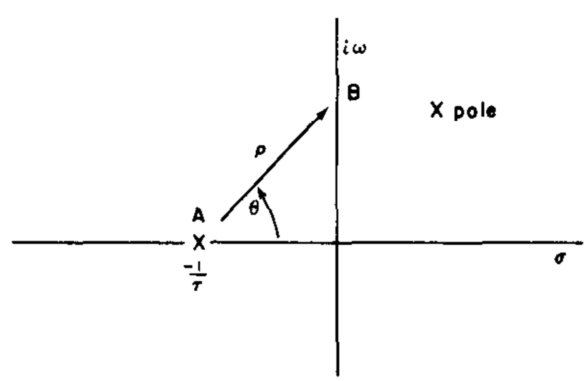

FIG. 2.—S plane representation of a low-pass filter.

Expressing the vector $\overline{AB}$ in its polar form, and since $T(s) = \dfrac{1}{\overline{AB}}$, we have $T(s) = \frac{e^{-i\theta}}{\rho}$. Hence, $A(\omega)$ behaves as $\frac{1}{\rho}$ and $\phi(\omega)$ as $-\theta$.

In other words, $\lim_{\omega \to 0} A(\omega) = [A(\omega)]\max$ and

$\lim_{\omega \to \infty} A(\omega) = 0$. Similarly, $\lim_{\omega \to 0} \phi(\omega) = 0$ and $\lim_{\omega \to \infty} \phi(\omega) = -\frac{\pi}{2}$. We know from experience that a low-pass filter does have these characteristics.

The response of the low-pass filter to an impulsive input may be conveniently obtained by returning to the definition of the transfer function.

$$T(s) = \frac{E_o(s)}{E_i(s)} = \frac{1}{1+s\tau}$$

where $E_o(s)$ and $E_i(s)$ are transforms of the output and input, respectively. In the case of an impulsive input,

$$E_i(s) = 1 \text{ and } E_o(s) = \frac{1}{1+s\tau}$$

$E_o(s)$ is by definition the transform of the output $e_o(t)$; thus $e_o(t)$ is the inverse transform of $\frac{1}{1+s\tau}$ or

$$e_o(t) = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) \quad \text{for } t>0$$

In other words, the output in time is an exponentially decaying transient.

At this point we wish to show that the transfer function of a system with a pole in the right-half plane (Fig. 3) is not physically realizable.

Proceeding as before,

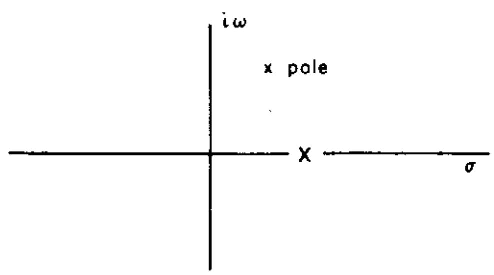

FIG. 3.—S plane representation of a pole in the right-half plane.

$$E_0(s) = \frac{1}{1 - s\tau}$$

and

$$e_0(t) = -\frac{1}{\tau}\exp\left(\frac{t}{\tau}\right) \qquad \text{for } t>0$$

We see therefore that the output increases indefinitely. It is clear that such behavior is not physically possible and hence we may generalize and state that the transfer function of a physically realizable system does not have poles in the right-half of the complex plane.

Let us now use the S plane representations of systems whose transfer functions have both a pole and a zero, to describe the concept of minimum-phase. Since we need not consider poles in the right-half plane the possible pole-zero configurations for such systems are shown in figure 4.

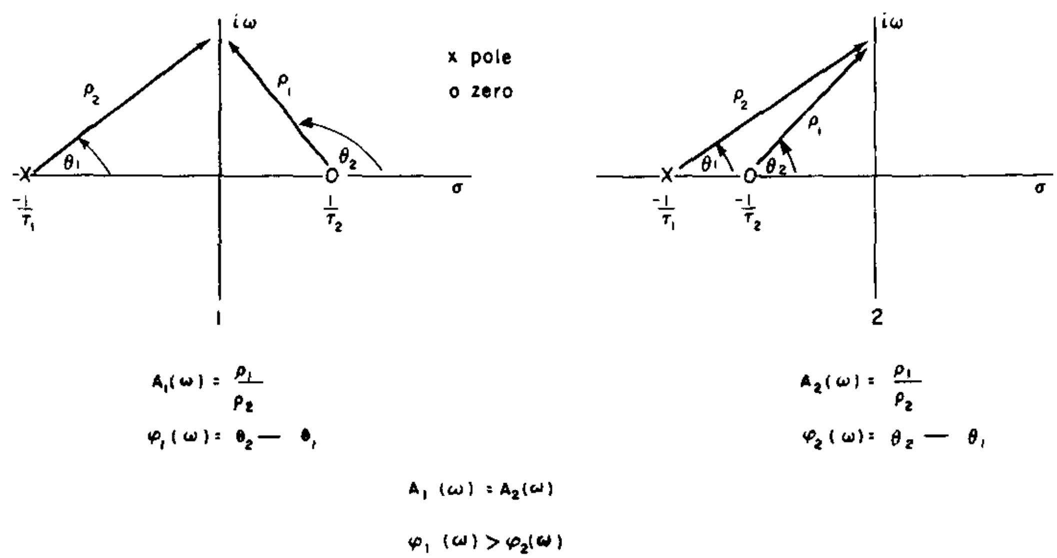

FIG. 4.—The pole-zero configuration for two circuits having the same amplitude response but different phase response.

It is clear from the preceding discussion and figure 4 that both circuits with the S plane configurations shown have the same amplitude response, but different phase response. In fact the circuit with all its poles and zeros in the left-half plane has the lesser phase response.

In general terms, if a number of circuits have the same amplitude response, that circuit which has all the poles and zeros of its transfer function in the left-half complex plane, is the *minimum-phase* circuit.

CAUSALITY AND PHYSICAL REALIZABILITY

It is obvious that all physically realizable systems must be causal. In other words their output cannot precede their input.

We have found that the question, "Are non-minimum-phase filters physically realizable?" is often asked. We will therefore develop two conditions which must be satisfied for causality to hold, and we will then apply them to a non-minimum-phase filter.

In order to develop these conditions we introduce the impulse response of a system I(t). I(t) is real and may be written as

$$I(t) = E(t) + O(t) \tag{1}$$

where E(t) and O(t) are the even and odd parts, respectively.

$T(i\omega)$ and I(t) are related through the Fourier transform

$$T(i\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} I(t)\, e^{-i\omega t}\, dt = \frac{1}{\pi}\int_{o}^{+\infty} E(t)\, \cos \omega t\, dt$$

$$- \frac{i}{\pi}\int_{o}^{+\infty} O(t)\, \sin\omega t\, dt \tag{2}$$

Also, from our previous discussion

$$T(i\omega) = G(\omega) + i\,B(\omega) \tag{3}$$

It follows from (2) and (3) that $G(\omega)$ is the Fourier transform of E(t) and $B(\omega)$ is the Fourier transform of O(t).

We will choose to write equation (1) as

$$I(t) = E(t) + a\delta(t) + O(t) \tag{4}$$

where $\delta(t)$ is the delta function or the unit impulse and a is a constant which may be zero. $\delta(t)$ is an even function and we have extracted it from the even part of equation (1). The reason for this choice will become clear later.

Since for physically realizable systems there can be no output before there is an input,

$$I(t) = o \text{ for } t < o \tag{5}$$

We define a function

$$\operatorname{sgn} t = \begin{cases} +1 & t>o \\ -1 & t<o \end{cases}$$

The Fourier transform of sgn t may be easily shown to be $\frac{1}{i\omega\pi}$ (Bracewell 1965).

From equations (4) and (5) we may write

$$O(t) = \operatorname{sgn} t \, E(t) \qquad (6)$$

[The reason for our choice of equation (4) is now clear. $a\delta(t)$ has a value only at $t = o$, where sgn t is not defined. Consequently, $a\delta(t)$ is required so that circuits with impulsive behaviour at $t = o$ are included in our causality conditions.]

Substituting equation (6) into equation (4) and taking the Fourier transform of both sides

$$T(i\omega) = G(\omega) + \frac{1}{i\omega\pi} * G(\omega) + a \qquad \text{Where } * \text{ implies convolution}$$

$$= G(\omega) + a + \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{G(\omega')\, d\omega'}{\omega' - \omega}$$

Hence Condition I.

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{G(\omega')\, d\omega'}{\omega' - \omega} \dagger \qquad (7)$$

(where we take the Cauchy principal value of the integral.)

We may also write

$$E(t) = \operatorname{sgn} t \, O(t)$$

Hence

$$T\ (i\omega) = \frac{1}{i\omega\pi} * i\,B(\omega) + iB(\omega) + a$$

$$= -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{B(\omega')\, d\omega'}{\omega' - \omega} + a + i\,B(\omega)$$

†This integral formulation is known as the Hilbert transform.

Condition II

$$G(\omega) = a - \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{B(\omega')\ d\omega'}{\omega' - \omega} \tag{8}$$

Conditions I and II may be derived by contour integration (Tuttle 1958, Papoulis 1962) and it may be shown that the constant a is in fact $G(\infty)$.

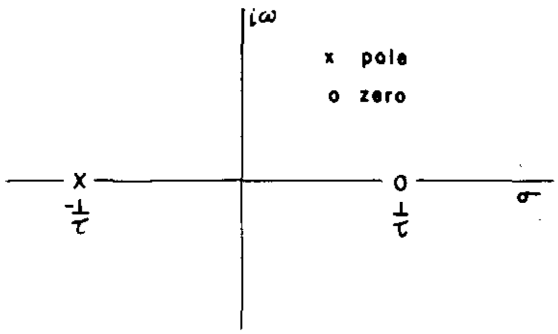

FIG. 5.—Pole-zero diagram of a non-minimum-phase system.

Let us apply the causality conditions to a non-minimum-phase circuit. Following our definition of minimum-phase, the transfer function of such a system may be represented in the S plane as shown in figure 5. This pole-zero pattern actually corresponds to the all-pass filter shown in figure 6.

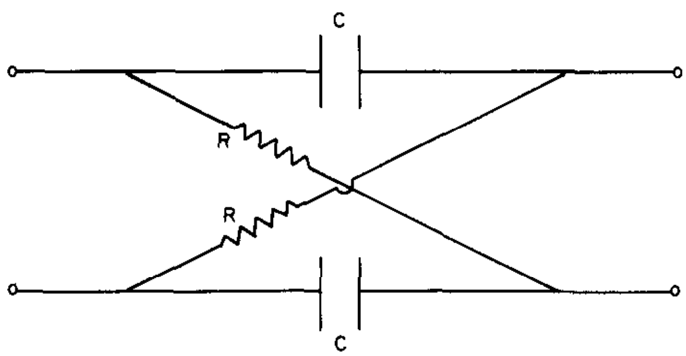

FIG. 6.—An all-pass filter.

For this circuit

$$T(s) = \frac{1 - s\tau}{1 + s\tau} \tag{9}$$

Hence, separating T(s) into its real and imaginary parts:

$$G(\omega) = \frac{1-\omega^2\tau^2}{1+\omega^2\tau^2}$$

and

$$B(\omega) = \frac{-2\omega\tau}{1+\omega^2\tau^2}$$

We may determine the impulse response for this system from equation (9) using the Laplace transform.

$$T(s) = -1 + \frac{2}{1+s\tau}$$

Hence

$$I(t) = -\delta(t) + \frac{2}{\tau}\exp\left(-\frac{t}{\tau}\right)$$

and is shown in figure 7.

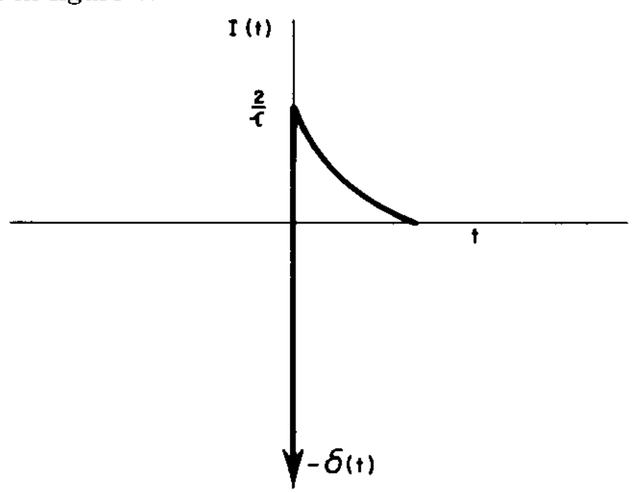

FIG. 7.—The impulse response of the all-pass filter.

The constant a in equation (8) is therefore —1. (Note that $G(\infty)$ is also —1).

We have two conditions to test.

$$\text{I.} \quad \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{G(\omega')\,d\omega'}{\omega'-\omega} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{-1+2/(1+\omega'^2\tau^2)\,d\omega'}{\omega'-\omega}$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{-\,d\omega'}{\omega' - \omega} + \frac{1}{\tau^2\pi}\int_{-\infty}^{+\infty} \frac{2\,d\omega'}{(\omega' + i/\tau)(\omega' - i/\tau)(\omega' - \omega)}$$

We solve both integrals by the Cauchy principal value theorem. The *first integral may be easily shown to be zero. The second integral is* equal to $2\pi i\,\Sigma$ residues at poles in upper half of complex plane plus $\pi i\,\Sigma$ residues on real axis.

Consequently the second integral is

$$\frac{2\pi i\,(2\tau)}{\tau^2\pi(2i)\,(i/\tau - \omega)} + \frac{\pi i(2)}{\tau^2\pi(\omega + i/\tau)\,(\omega - i/\tau)}$$

$$= \frac{-\,2\omega\tau}{1 + \omega^2\tau^2} = B(\omega)$$

Condition I is therefore satisfied.

II. We must find the value of

$$-1 - \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{-\,2\omega'\tau\,d\omega'}{(1 + \omega'^2\tau^2)\,(\omega' - \omega)}$$

Proceeding as above, we have the principal value

$$-1 - \frac{2\pi i}{\pi\tau(i/\tau - \omega)} + \frac{\pi i(2\omega)}{\pi\tau(\omega + i/\tau)(\omega - i/\tau)}$$

$$= -1 + \frac{2}{1 + \omega^2\tau^2}$$

$$= \frac{1 - \omega^2\tau^2}{1 + \omega^2\tau^2} = G\ (\omega)$$

Condition II is also satisfied.

We may state therefore that causality does not imply minimum-phase. The poles and zeros of figure 5 correspond to a physically realizable filter which is not a minimum-phase system.

Thus, the only condition which physically realizable passive systems must satisfy is that their transfer functions do not possess poles in the right-half of the complex plane.

Why, then, are we concerned with minimum-phase systems at all? We will answer this question only in part here.

$$T(i\omega) = G(\omega) + i\,B(\omega)$$
$$= A(\omega)\ \exp\ i\Phi(\omega)$$

Hence

$$\log_e T(i\omega) = \log_e A(\omega) + i\Phi(\omega)$$

If we identify $\log_e A(\omega)$ with $G(\omega)$

and $\Phi(\omega)$ with $B(\omega)$

we may write, using Condition II

$$\Phi(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\log_e A(\omega')\, d\omega'}{\omega' - \omega} \tag{10}$$

for a causal system. We may therefore determine the phase response from the amplitude response, a procedure often desirable.

It may easily be shown that a system with a pole in the right-half plane violates the causality conditions (as it must, since we have already seen that such systems are not physically realizable).

But $\log_e T(i\omega)$ has a pole in the right-half plane for every zero of $T(i\omega)$ in the right-half plane. Consequently $\log_e T(i\omega)$ will violate the causality conditions unless we specify that all the zeros of $T(i\omega)$ be confined to the left-half plane. In other words, for equation (10) to hold, $T(i\omega)$ must correspond to a minimum-phase system.

Minimum phase is also related to stability of digital filters and deconvolution. We refer the reader to excellent discussions of these topics by Treitel and Robinson (1964), Rice (1962), Robinson (1966), and Ford and Hearne (1966).

SUMMARY

We have shown that the condition of causality, in other words that the output does not precede the input, implies that the transfer function of a physically realizable passive system does not possess poles in the right-half of the complex plane. This is not sufficient to specify a minimum-phase system which also must not possess zeros in the right-half plane.

We have also shown that the phase response of a system may be obtained from a knowledge of the amplitude response only for a minimum-phase system.

We have reviewed these matters here because we believe it is of importance that the concept of minimum-phase be dealt with from basic considerations in the geophysical literature.

For additional discussion we refer the reader to Bode (1945) and Papoulis (1962).

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