SOME REMARKS CONCERNING THE BOTT-SMITH INEQUALITIES FOR DEPTH DETERMINATIONS OF GRAVITATING BODIES

By

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ABSTRACT

The Bott-Smith inequalities were originally formulated to obtain the limiting depths to the tops of gravitating bodies of arbitrary shape. In each case the true depth is less than or equal to the value computed, and in this regard the inequalities appear to work well. However, in the oil or mining industries we are concerned with pinpointing the structures under investigation. Thus we are mainly interested in the equality sign. The question then arises as to how we can best use these expressions, or perhaps modify them, to give us good, reliable depth estimates that have real significance. In order to help answer this the gravity effect of various well known models were investigated and the Bott-Smith relations computed on a theoretical basis.

METHOD

Four of the Bott-Smith inequalities (1) were investigated and the particular notation used to distinguish them was:

For 2-Dimensional Bodies

$$D2 \leqslant \frac{g_x}{\frac{dg_x}{dx}}$$

For 3-Dimensional Bodies

$$D3 \leqslant \frac{1.5\, g_x}{\frac{dg_x}{dx}}$$

$$D4 \leqslant \frac{3\sqrt{3}}{8} \frac{g_{max}}{\left|\frac{dg}{dx}\right|_{max}}$$

$$D9 \leqslant \frac{48\sqrt{5}}{125} \cdot \frac{g_{max}}{\left|\frac{dg}{dx}\right|_{max}}$$

For the 2-D case a horizontal cylinder and a horizontal half plate were used and for the 3-D case a sphere and a circular disc were used. These models were chosen because their gravity expressions are relatively simple and one might think that the above inequalities would work best on such simple models.

The expressions for the anomalous gravity effect of these four models have been derived by Nettleton (2) and (3). The theoretical expression for the gradient of each curve was obtained by differentiation. The maximum value of the gradient was then obtained by further differentiation to find the value of x at which the gradient was a maximum, and

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this was substituted in the expression for the gradient to give $\left|\frac{dg}{dx}\right|_{max.}$

A summary of these results is given in Appendix I, and the expressions are such that g is given in milligals and all distances are in kilofeet.

Simple programs were written to calculate these values for the four models at various depths and the Bott-Smith depth estimates were obtained.

RESULTS

(a) For the *sphere* and the *circular disc* the D9 formula gave the depth to the centre of these bodies exactly.

(b) For the *cylinder* the D4 formula gave the depth to the centre of this body exactly.

(c) For the *sphere, circular disc* and *cylinder* the following was noted:

—When $\frac{x}{z} = 1$, the D2 or D3 formulas gave the depth to the centre;

—When $0.7 < \frac{x}{z} < 1.6$ the D2 or D3 formulas gave the depth to the centre to within 10% error.

These can be summarized with the following diagram:

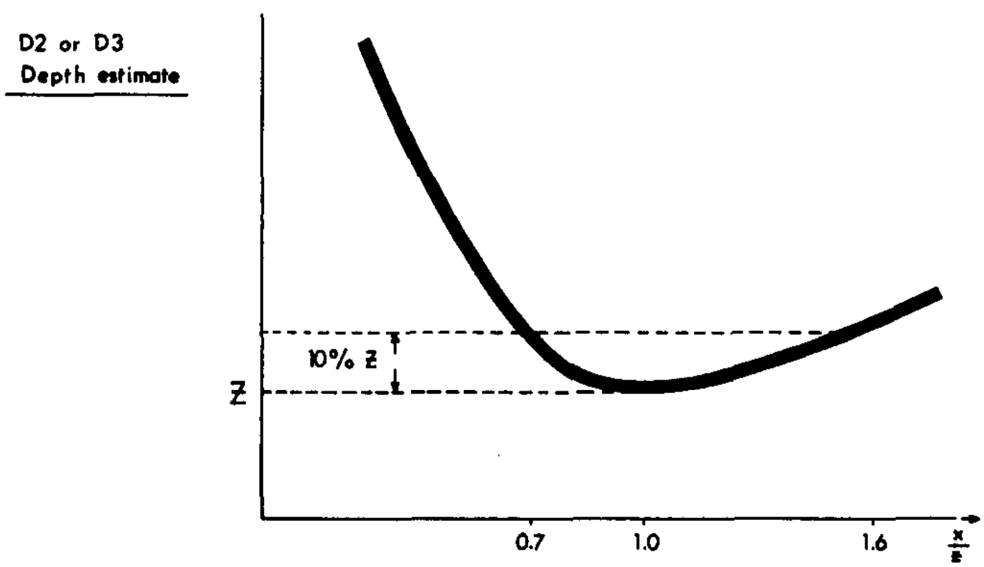


FIG. 1.

(d) For the *horizontal half plate* all the depth estimates obtained were too large to be of real value.

DISCUSSION

In deriving the expression for the gravity anomaly of a sphere we make use of a result of potential theory, namely that the external effect of such a body is identical to that of a particle of equal mass at the centre. Similarly, for a horizontal cylinder of infinite length the external effect can be shown to be equivalent to a linear concentration of mass along its axis. The expressions for the gravity anomalies of the circular disc and horizontal half plate as quoted in Appendix I are only approximate in that the thickness of these bodies should be less than half their mean depths in order to keep the error to within a few per cent. However, providing that the thickness is only a fraction of the depth of burial, which is assumed in the derivation of the gravity effects, these expressions are exact.

It thus seems logical that the Bott-Smith inequalities should have given the depth to the centre in the case of the sphere, cylinder and circular disc because as Bott and Smith point out in their paper "all these inequalities are 'best possible' in the sense that the equality sign can actually be attained in cases where the body degenerates into a single particle in the plane z = -h." However, we need to elaborate upon the horizontal half plate because the depth estimates were not satisfactory for our needs.

The gravity anomaly for the horizontal semi-infinite half plate is given by

$$g = 2G\rho t\left(\frac{\pi}{2} \pm \tan^{-1}\frac{X}{Z}\right)$$

and using this we get

$$Z = \frac{1}{\pi} \cdot \frac{g_{max}}{\left|\frac{dg}{dx}\right|_{max}} \qquad \text{(See Appendix III)}$$

Thus in order to get the depth to the centre of this body we should replace $\frac{3\sqrt{3}}{8}$ by $\frac{1}{\pi}$ in the expression for D4.

We also note that $$\frac{g_x}{\frac{dg_x}{dx}} = Z\left(\frac{\pi}{2} \pm \tan^{-1}\frac{X}{Z}\right)\left[1 + \left(\frac{X}{Z}\right)^2\right]$$

The minimum value which this can have occurs when

$$\frac{X}{Z} = \tan\left(\frac{\pi}{2} - \frac{Z}{2X}\right)$$

and to satisfy this condition, $\frac{x}{z} = 0.43$.

Using this we get $$\frac{g_x}{\frac{dg_x}{dx}} = 1.37 \; Z$$

Thus when using the 2D formula, the nearest we will come to getting a depth estimate on the horizontal half plate is 1.37 times the depth to the centre and this occurs when x is 0.43 times the depth to the centre on the side of the anomaly which is "off" the half plate.

It must be admitted that this model has limited use and a more versatile formula is the one derived in Appendix II for the buried step, with no limitations on the values of h and t. Similar comments regarding the Bott-Smith D2 formula will apply to this model and in order to get around this Bancroft (4) gave a formula for determining the depth to the top of such a buried step (equation (D) of Appendix III), and this formula works excellently on all types of such models from large intra-basement features to near surface features. Appendix III is given for the sake of completeness with apologies to Bancroft.

When considering the horizontal slab with sloping interface (Appendix IV) it has been noted by Geldart et al (5) that providing the dip of the fault plane truncating the slab lies between 90° and 15°, which would include most practical cases, then we get only a small change in curve shape, and in particular only a small change in the maximum gradient value. Thus, perhaps the best quick estimate of the depth to

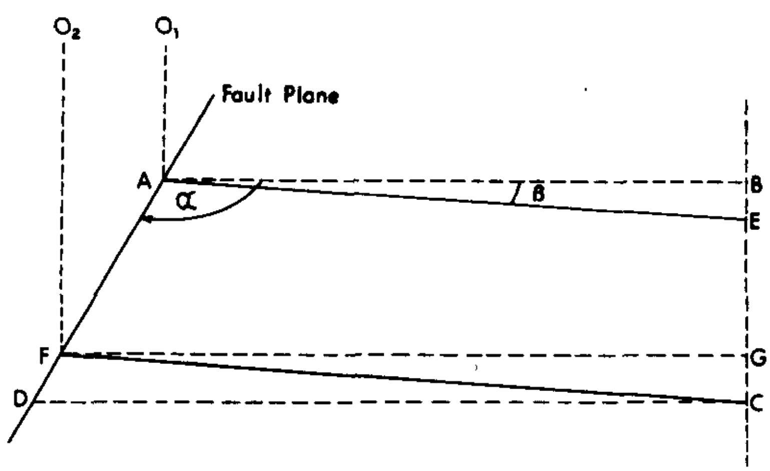


FIG. 2.

the top of such a body will be given by using Bancroft's formula and assuming the interface to be vertical, because g max will remain the same and U_{xz} will only change slightly and so d_0 (equation (C)) will be only slightly different.

The importance of equation (12) of Appendix IV is that it allows us to model most two dimensional fault problems fairly easily. Consider for example fig. 2.

By calculating the gravity effect of the slab ABCD using angle α and subtracting that due to ABE using β we get the effect of body AECD, and this would be plotted using 0_1 as origin. By obtaining the effect of slab FGCD using α again and subtracting the effect of FGC using β we get the effect of body FCD and this would be plotted using 0_2 as origin. By subtracting these two curves we would be left with the effect of body AECF which is a slab dipping at angle β and truncated with a fault dipping at $180° - \alpha$ in the opposite direction.

Now consider fig. 3.

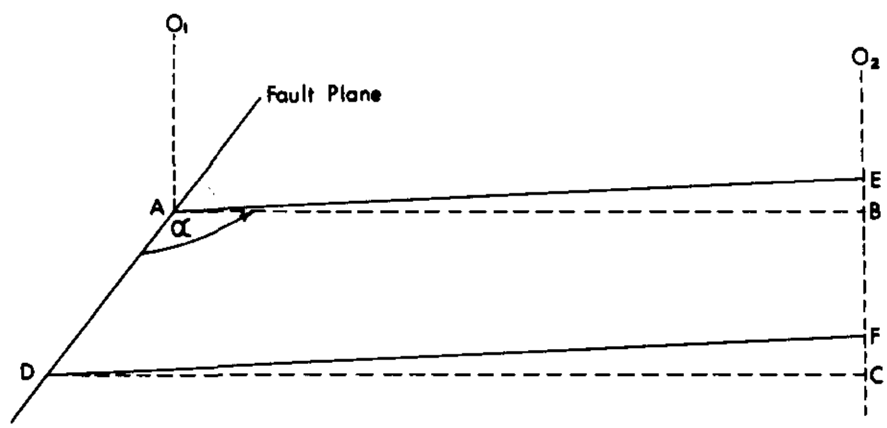


FIG. 3.

Here we would calculate the gravity effect of slab ABCD using α and plot this using 0_1 as origin. Then we would calculate the effect of the wedge ABE and subtract the effect of wedge DFC and then plot this curve using 0_2 as origin. By adding these two curves we would get the effect of slab AEFD. The gravity effect of such a wedge as AEB is readily obtained from Appendix IV by integrating in the x direction from zero to $(z - h) \cot\alpha$ instead of from $(z - h) \cot\alpha$ to infinity.

For any such model care must be taken to extend the model well outside the area of interest in order to eliminate slight edge effects which will otherwise occur.

A NOTE ON THE USE OF THE BOTT-SMITH RELATIONSHIPS TO SEPARATE OUT INTERFERING ANOMALIES

If we have a good clean anomaly due to a body which can be approximated to a horizontal cylinder we have the following information available to us in order to calculate the depth to the centre (z):

(a)	*Half-width*	at the position where the gravity value is half its maximum value, $x = z$
(b)	*Maximum Gradient*	at this point $x = 0.577\ z$
(c)	*D2 Value*	at $x = z$ this value equals z
(d)	*D4 Value*	this equals z

In the case of two such bodies occurring near and parallel to each other, their anomalies will interfere and all of these depth estimates will be invalidated. A method is now proposed whereby we can not only obtain a good estimate of the depth to the centre of these bodies, but whereby we can also separate out their respective anomalies.

Figure 4 shows the synthesis of two anomalies coming from two cylinders of material of density contrast 0.2 gm./c.c., buried to a depth of 10,000 feet and of 1,000 feet radius.

Considerable error will result in making the half-width measurement at the wrong place and we end up with wildly incorrect results. The suggestion is that by making several independent depth estimates, such as the half width, the D2 value at $\frac{x}{z} = 1$ and the D4 value, all of which we know should be consistent, then, by a reiteration process, we should be able to reconstruct the original anomalies by obtaining a consistent set of depth estimates at the optimum position.

By way of illustration consider Figures 4 and 5. We assume that we know from the geology of the area that we are dealing with 2-dimensional bodies which can be approximated to cylinders.

Suppose we were analyzing the theoretical field data curve of Figure 4. This is reproduced again in Figure 5. By taking this data and using profile smoothing we might have placed the base line at 1.572 g.u. This would have resulted in the following measurements:

Half Width

$$g\ \text{max.} = 2{,}809 - 1.572 = 1.237\ \text{g.u.}$$
$$\tfrac{1}{2}\ g\ \text{max.} = 0.619\ \text{g.u.}$$
$$x\ \tfrac{1}{2} = z = 6{,}470\ \text{ft.}$$

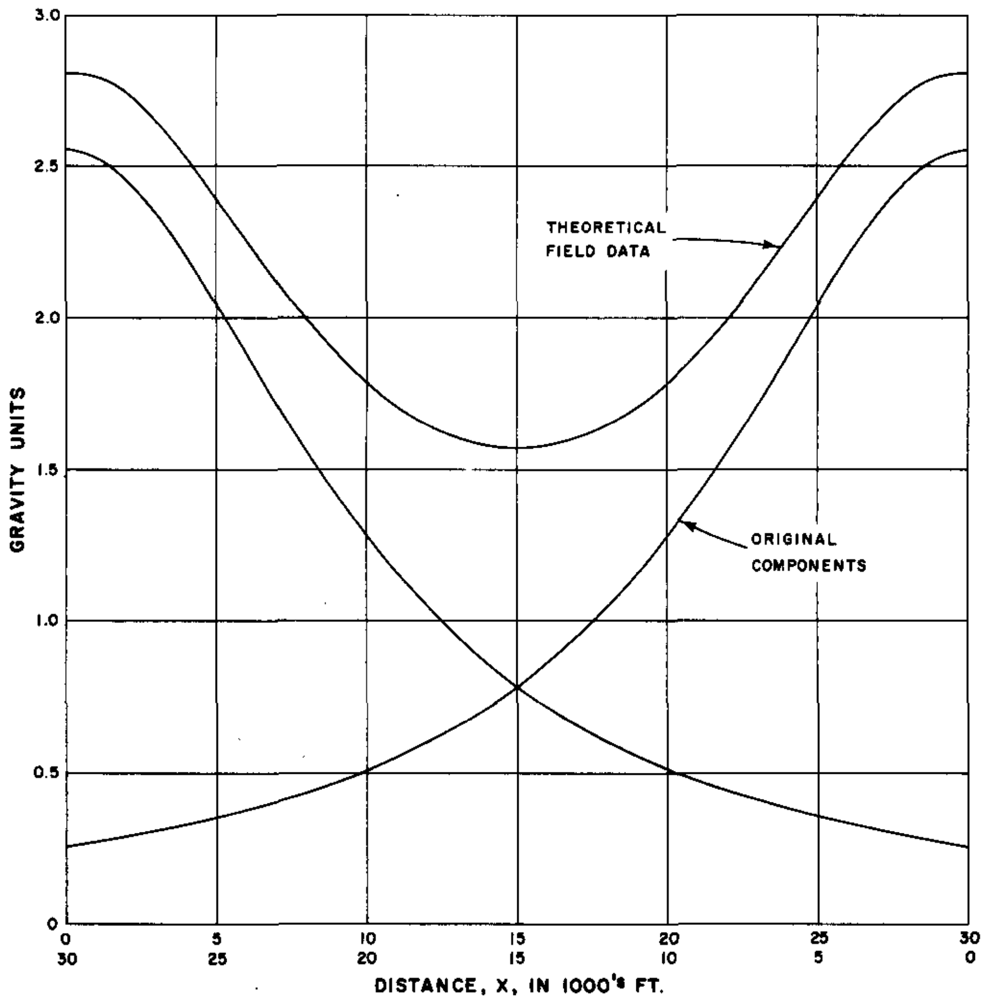


FIG. 4.

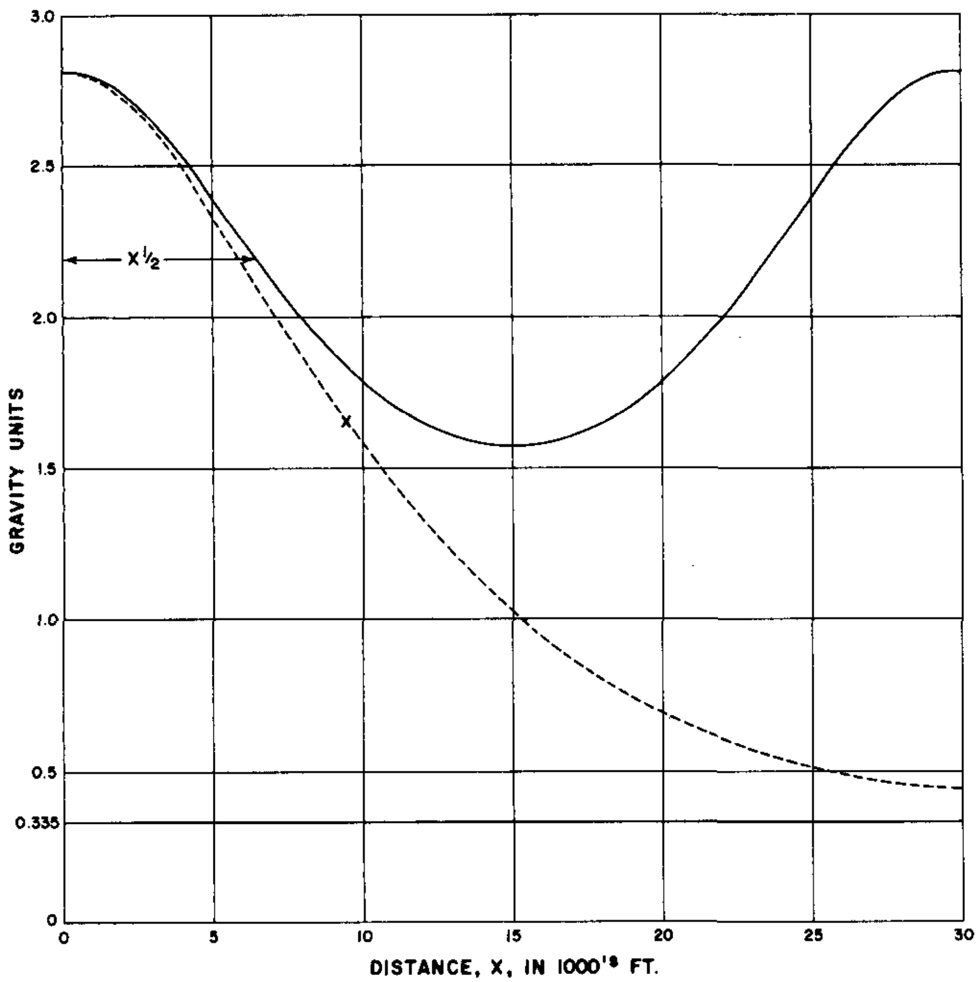
Maximum Gradient

We know that at the point of maximum gradient,

$$\frac{x}{z} = 0.577$$

measuring x at this point we obtain x = 5,450 feet

$$\text{i.e.}\quad Z = \frac{5,450}{0.577} = 9,450 \text{ feet}$$



RECONSTRUCTION

FIG. 5

These two values should agree exactly. The fact that they do not is suggestive of interference of adjacent anomalies. We know that the maxima of the anomalies will be least affected by adjacent ones, and in this example from 0 to 8,000 feet we are adding quantities from the interfering anomaly which lie on an approximate straight line. Thus the position of the point of maximum gradient will be only little affected in comparison with the half-width position. Thus we would expect the maximum gradient value to be the better of the two. Using this value, and knowing that we need to take the D2 measurement where

$\frac{x}{z} = 1$, we use x = 9,450 feet.

$$\text{Then, } D2 = \frac{g}{\left|\frac{dg}{dx}\right|} = \frac{0.265}{9.55} \times 10^5 = \underline{\underline{2,880 \text{ feet}}}$$

$$\text{Also } D4 = \frac{0.6495 \text{ g max.}}{\left|\frac{dg}{dx}\right|_{max.}} = \frac{0.6495 \times 1.237}{1.408 \times 10^{-4}} = \underline{\underline{5,700 \text{ feet}}}$$

The inconsistency in these 4 values tells us that, if we are dealing with a 2-dimensional body which can be approximated to a cylinder, then there is something drastically wrong.

The particularly low value for D2 is interesting. This implies that the value of g used is too low. This is also supported by the low value for D4, impying that g max. is too small. Thus, as previously mentioned, $\left|\frac{dg}{dx}\right|_{max.}$ *will not change very much, so that the D4 expression* can be used to obtain a rough estimate of g max. The first thing one would do here is to double g max. used above; i.e. to *2.476 g.u.* Therefore we draw a new base line at *0.335 g.u.*

We also know that g used above in D2 can be increased about 5 times (assuming that the gradient is going to be decreased slightly); i.e. to 0.265 x 5 = *1.325 g.u.* This gives us a fix from the new base line and is marked with an X.

Thus we would draw in the dotted line as shown and repeat all four measurements. Note now that the new half-width measurement would be made at 1.572 g.u. and

$$x_{1/2} = z = 10{,}100 \text{ feet}$$

The maximum gradient value will be approximately the same as before, and the new D2 and D4 values will be in fairly close agreement, *simply by virtue of the reconstruction.*

A second reiteration could now be made to perfect the agreement between the four values, if considered necessary.

CONCLUSION

Making a simple half-width measurement on the original field data gives a hopelessly inadequate result. Using this technique, after only one iteration we have arrived at a good estimate for the depth of burial and a fair separation of the component anomalies. If density information is available (e.g. a nearby density log), then the radius could be estimated using

$$g\ max. = \frac{12.77\rho R^2}{Z}$$

$$i.e.\ R = \sqrt{\frac{Z\ .\ g\ max.}{12.77\rho}}$$

REFERENCES

(1) BOTT, M. H. P. and SMITH, R. A., 1958: The Estimation of the Limiting Depth of Gravitating Bodies, Geop. Pros. Vol. VI, No. 1, pp. 1-10.

(2) NETTLETON, L. L., 1940: Geophysical Prospecting for Oil, McGraw-Hill.

(3) NETTLETON, L. L.: Gravity and Magnetic Calculations, Geophysics, Vol. VII, No. 3, pp. 293-310.

(4) BANCROFT, A. M.: Gravity Anomalies over a Buried Step, Jour. Geop. Research Vol. 65, pp. 1630-31, 1960.

(5) GELDART, L. P., DENIS E. GILL, and BIJON SHARMA, 1966; Gravity Anomalies of Two-Dimensional Faults, Geophysics, Vol. XXXI, No. 2, pp. 372-397.

(6) GRANT and WEST: Interpretation Theory in Applied Geophysics, McGraw-Hill, 1965.

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APPENDIX I

Cylinder

$$g = 12.77\frac{\rho R^2}{Z}\frac{1}{\left[1+\left(\frac{X}{Z}\right)^2\right]}$$

$$\left|\frac{dg}{dx}\right| = 25.54\frac{\rho R^2}{Z^2}\frac{\frac{X}{Z}}{\left[1+\left(\frac{X}{Z}\right)^2\right]^2}$$

max. occurs at $\frac{X}{Z} = \pm \frac{1}{\sqrt{3}} = \pm 0.577$

$$\left|\frac{dg}{dx}\right|_{max.} = 8.293 \frac{\rho R^2}{Z^2}$$

Horizontal Half Plate

$$g = 4.064 \rho t \left(\frac{\pi}{2} - \tan^{-1} \frac{X}{Z} \right)$$

$$\left|\frac{dg}{dx}\right| = 4.064 \frac{\rho t}{Z} \frac{1}{\left[1 + \left(\frac{X}{Z}\right)^2\right]}$$

max. occurs at $\frac{X}{Z} = 0$

$$\left|\frac{dg}{dx}\right|_{max.} = 4.064 \frac{\rho t}{Z}$$

ρ is the density contrast in gm./cc.

R is the radius.

Z is the depth of burial to the centre of the body.

t is the thickness of the half plate.

x is the horizontal distance from the centre of the anomaly.

Sphere

$$g = 8.513\ \frac{\rho R^3}{Z^2}\ \frac{1}{\left[1+\left(\frac{X}{Z}\right)^2\right]^{3/2}}$$

$$\left|\frac{dg}{dx}\right| = 25.539\ \frac{\rho R^3}{Z^3}\ \frac{\frac{X}{Z}}{\left[1+\left(\frac{X}{Z}\right)^2\right]^{5/2}}$$

max. occurs at $\frac{X}{Z} = \pm\ 1/2$

$$\left|\frac{dg}{dx}\right|_{max.} = 7.309\ \frac{\rho R^3}{Z^3}$$

Circular Disc

$$g = 2.032\ \rho\,\Omega\,t. = 2.032\ \frac{\rho t s}{Z^2}\ \frac{1}{\left[1+\left(\frac{X}{Z}\right)^2\right]^{3/2}}$$

$$\left|\frac{dg}{dx}\right| = 6.096\ \frac{\rho t s}{Z^3}\ \frac{\frac{X}{Z}}{\left[1+\left(\frac{X}{Z}\right)^2\right]^{5/2}}$$

max. occurs at $\frac{X}{Z} = \pm\ 1/2$

$$\left|\frac{dg}{dx}\right|_{max.} = 1.745\ \frac{\rho t s}{Z^3}$$

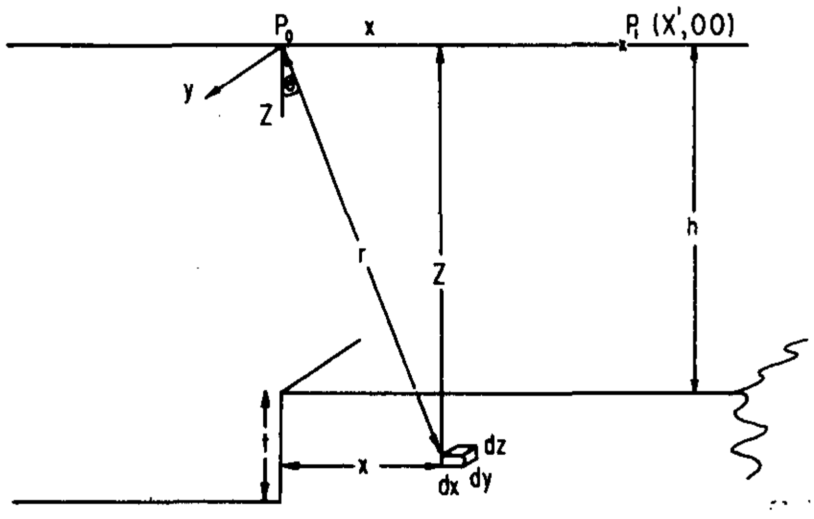
Here, Ω is the solid angle subtended by the *surface* of the disc.

t is the thickness of the disc.

s is the surface area of the disc.

APPENDIX II

DERIVATION OF A RIGOROUS EXPRESSION FOR THE GRAVITY ANOMALY OF A BURIED STEP



Consider a slab of material infinite in the y direction, of thickness t, with depth to the top equal to h, and density contrast with surrounding material equal to ρ. Consider further an elemental volume of this slab dxdydz at depth z and at distance x from the edge of the slab, and take the point P_0 at the surface to be at the origin of our coordinate system.

The anomalous gravitational force at P_0 due to this elemental volume will be:

$$\Delta F = m\Delta g = \frac{GmM\cos\theta}{r^2}$$

where G is the gravitational constant, and M is the mass of the elemental volume.

When g is in gals, m is unity and we have:

$$M = \rho\, dx\,dy\,dz\,; \qquad \cos\theta = \frac{Z}{r}\,; \qquad r^2 = z^2 + x^2 + y^2\,;$$

so $$\Delta g = \frac{G\rho z.\,dx\,dy\,dz}{\left[x^2 + y^2 + z^2\right]^{3/2}}$$

To obtain the total anomalous gravity value at P due to the entire slab we integrate z between the limits h and h+t, y between the limits $-\infty$ to $+\infty$ and x from 0 to $+\infty$. i.e.

$$g = G\rho \int_{x=0}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=h}^{h+t} \frac{z\,dx\,dy\,dz}{\left[x^2+y^2+z^2\right]^{3/2}}$$

$$= G\rho \int_0^{\infty}\int_{-\infty}^{\infty}\left[-\frac{1}{\{x^2+y^2+z^2\}^{1/2}}\right]_h^{h+t} dx\,dy$$

$$= G\rho \int_0^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{\{x^2+y^2+h^2\}^{1/2}} - \frac{1}{\{x^2+y^2+(h+t)^2\}^{1/2}}\right] dx\,dy$$

For y the integral from $-\infty$ to $+\infty$ will be the same as twice that from 0 to $+\infty$; i.e.

$$g = 2G\rho \int_0^{\infty}\left[\ln\left\{y + \sqrt{y^2 + x^2 + h^2}\right\} - \ln\left\{y + \sqrt{y^2 + x^2 + (h+t)^2}\right\}\right]_0^{\infty} dx$$

$$= 2G\rho \int_0^{\infty}\left[\ln\sqrt{x^2 + (h+t)^2} - \ln\sqrt{x^2 + h^2}\right] dx$$

$$= G\rho \left[\int_0^{\infty} \ln\{x^2 + (h+t)^2\}\,dx - \int_0^{\infty} \ln(x^2 + h^2)\,dx\right]$$

Having integrated in the z and y directions, consider the point P_1 $(x', 0, 0)$ instead of P_o $(0, 0, 0)$. We have to do this in order to end up with a general expression for the gravity value at all values of x'. If we merely carry out the above integration we will end up with the value at P only. This now means that we will have to change the lower integration limit from 0 to $-x'$. i.e.

$$g = G\rho\left[\int_{-x'}^{\infty} \ln\{x^2 + (h+t)^2\}\,dx - \int_{-x'}^{\infty} \ln(x^2 + h^2)\,dx\right]$$

$$= G\rho\Big[x\ln\{x^2 + (h+t)^2\} - 2x + 2(h+t)\arctan\frac{x}{h+t}$$

$$- x\ln(x^2+h^2) + 2x - 2h.\arctan\frac{x}{h}\Big]_{-x'}^{\infty}$$

$$= G\rho\left[\pi t + x'\ln\left\{\frac{x'^2+(h+t)^2}{x'^2+h^2}\right\} + 2(h+t)\tan^{-1}\frac{x'}{h+t} - 2h.\tan^{-1}\frac{x'}{h}\right]$$

APPENDIX III

Derivation of Bancroft's Formula for determining the Depth to the Top of a Buried Step.

Using the rigorous derivation for the anomalous gravity values associated with a buried step, and taking the origin as being directly over the vertical contact, we have the following:

$$g = G\rho\left[\pi t + x\ln\left\{\frac{x^2+(h+t)^2}{x^2+h^2}\right\} + 2(h+t)\tan^{-1}\frac{x}{h+t} - 2h.\tan^{-1}\frac{x}{h}\right]$$

When $x = 0$, $g = G\rho\pi t$.

Now g max. (the difference between the anomalous value at $x = +\infty$ and that at $x = -\infty$) will be double this. i.e.

$$\boxed{g\ max. = 2G\rho\pi t} \quad \text{-----------------(A)}$$

Differentiating the expression for g we get:

$$\frac{dg}{dx} = G\rho\left[\ln\left\{\frac{x^2+(h+t)^2}{x^2+h^2}\right\} + x\left\{\frac{x^2+h^2}{x^2+(h+t)^2}\right\}\left\{\frac{(x^2+h^2).2x - 2x\{x^2+(h+t)^2\}}{(x^2+h^2)^2}\right\} + \frac{2(h+t)^2}{(h+t)^2+x^2} - \frac{2h^2}{h^2+x^2}\right]$$

The maximum value of the gradient occurs when $x = 0$. i.e.

$$\left|\frac{dg}{dx}\right|_{max.} = G\rho\ln\left\{\frac{(h+t)^2}{h^2}\right\} = \boxed{2G\rho\ln\left(1+\frac{t}{h}\right) = U_{xz}} \quad \text{--(B)}$$

Now the expression for the gravity profile across a very thin horizontal half-plane whose thickness is L, and whose density contrast is $\Delta\rho$, and whose depth is z is given, at a horizontal distance x from its edge, as

$$g = 2G\Delta\rho L\left(\frac{\pi}{2} \pm \tan^{-1}\frac{x}{z}\right)$$

e.g. see Nettleton or Grant and West page 285.

Using this:

$$\frac{dg}{dx} = \pm\frac{2G\Delta\rho L}{z}\frac{1}{1+\left(\frac{x}{z}\right)^2}$$

The maximum value occurs when $x = 0$. i.e.

$$\left|\frac{dg}{dx}\right|_{max.} = \frac{2G\Delta\rho L}{z}$$

or $$z = \frac{2G\Delta\rho L}{\left|\frac{dg}{dx}\right|_{max}} = \frac{2\pi G\Delta\rho L}{\pi\left|\frac{dg}{dx}\right|_{max}} = \frac{1}{\pi}\frac{g\ max}{\left|\frac{dg}{dx}\right|_{max}}$$

Bancroft says let us put

$$\boxed{d_o = \frac{1}{\pi}\cdot\frac{g\ max.}{U_{xz}}} \qquad \text{------(C)}$$

The physical significance of this equation is that when the throw of the buried step is small compared with the depth of burial, then d will yield the depth to the centre of this slab. But usually the value is quite meaningless, however the right hand side of equation (C) is dimensionally correct. This enables us to use equations (A) and (B) to get an expression for h. viz:

$$d_o = \frac{1}{\pi}\frac{2\pi G\rho t}{2G\rho\ln\left(1+\frac{t}{h}\right)} = \frac{t}{\ln\left(1+\frac{t}{h}\right)}$$

$$\therefore \quad \ln\left(1+\frac{t}{h}\right) = \frac{t}{d_o}$$

$$\therefore \quad 1+\frac{t}{h} = e^{t/d_o}$$

$$\therefore \quad \boxed{h = \frac{t}{e^{t/d_o}-1}} \qquad \text{------(D)}$$

The way in which this equation can be used with field data is the following:

(a) g max. and U_{xz} are measured directly and ρ is either obtained from a density log or is estimated.

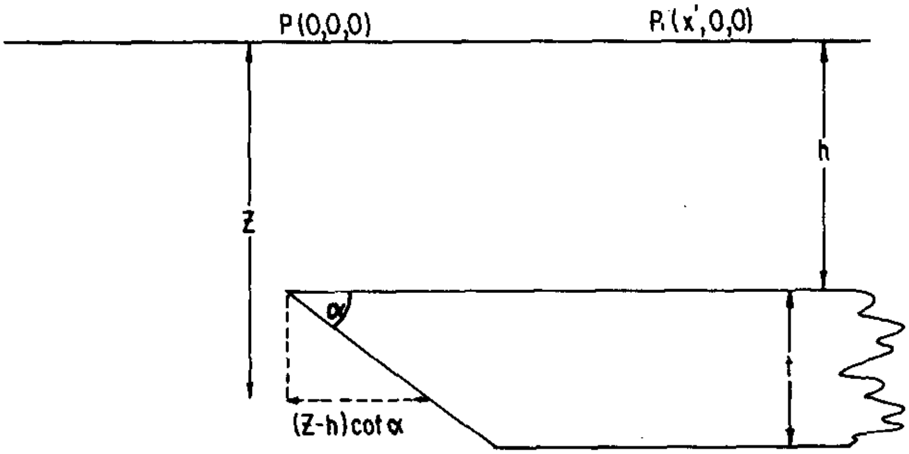
(b) t is obtained from (A).

(c) d_o is obtained from (C).

(h) *h is then calculated from (D).*

APPENDIX IV

Derivation of the Gravity Anomaly of a Horizontal Slab with Sloping Interface.



At the point P_1 (x′, 0, 0) the anomalous gravity value due to this slab of material of density contrast ρ is given by:

$$g(x') = \int_x \int_y \int_z \frac{G\rho z \, dx\,dy\,dz}{\left[(x-x')^2 + y^2 + z^2\right]^{3/2}} \qquad (1)$$

Integrating this expression in the y direction from $-\infty$ to $+\infty$ is equivalent to twice the integral from 0 to $+\infty$. Consider the integral:

$$\int_0^\infty \frac{dy}{(y^2 + a^2)^{3/2}}$$

Substituting $y = a.\tan\theta$, and $dy = a.\sec^2\theta d\theta$ we get

$$\int_0^{\frac{\pi}{2}} \frac{a.\sec^2\theta d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{1}{a^2}\left[\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{1}{a^2} \qquad (2)$$

Thus

$$g(x') = 2G\rho \int_x \int_z \frac{z\,dx\,dy}{(x-x')^2 + z^2} \qquad (3)$$

The limits of integration for x will be from (z-h) $\cot\alpha$ to $+\infty$, and as the variable z is included in the lower limit we must perform this operation next. i.e.

$$
\begin{aligned}
g(x') &= 2G\rho \int_h^{h+t} \left[\tan^{-1}\left(\frac{x-x'}{z}\right)\right]_{(z-h)\cot\alpha} dz \\
&= 2G\rho \int_h^{h+t} \left[\frac{\pi}{2} - \tan^{-1}\left\{\frac{(z-h)\cot\alpha - x'}{z}\right\}\right] dz \\
&= 2G\rho \int_h^{h+t} \left[\frac{\pi}{2} - \tan^{-1}\left\{ \cot\alpha - \frac{(h\cot\alpha + x')}{z}\right\}\right] dz \qquad (4)
\end{aligned}
$$

Let us consider

$$\int_h^{h+t} \tan^{-1}\left(a - \frac{b}{z}\right) dz \qquad \text{where } a = \cot\alpha \text{ and } b = h.\cot\alpha + x'$$

We start by noting that

$$\frac{d}{dz}\left[z \tan^{-1}\left(a - \frac{b}{z}\right)\right] = \tan^{-1}\left(a - \frac{b}{z}\right) + \frac{b/z}{1 + \left(a - \frac{b}{z}\right)^2}$$

$$\therefore \int \tan^{-1}\left(a - \frac{b}{z}\right) dz = z.\tan^{-1}\left(a - \frac{b}{z}\right) - b\int \frac{dz}{z\left[1 + \left(a - \frac{b}{z}\right)^2\right]} \qquad (5)$$

Now $$\int \frac{dz}{z\left[1 + a^2 - \frac{2ab}{z} + \left(\frac{b}{z}\right)^2\right]} = \int \frac{z\,dz}{b^2 - 2abz + (1+a^2)z^2}$$

This is of the form

$$\int \frac{z\,dz}{A + Bz + Cz^2} \qquad \text{where } A = b^2,\; B = -2ab,\; C = 1 + a^2$$

And this is equal to

$$\frac{1}{2C}\left[\ln(A+Bz+Cz^2)-\frac{2B}{\sqrt{4AC-B^2}}\tan^{-1}\frac{2Cz+B}{\sqrt{4AC-B^2}}\right]$$

Upon substituting for A, B and C this becomes

$$\frac{1}{2(1+a^2)}\left[\ln\left\{b^2-2abz+(1+a^2)z^2\right\}+2a.\tan^{-1}\left\{\left(\frac{1+a^2}{b}\right)z-a\right\}\right] \qquad (6)$$

Thus referring to (5) we get

$$\int_h^{h+t}\tan^{-1}\left(a-\frac{b}{z}\right)dz=\left[z.\tan^{-1}\left(a-\frac{b}{z}\right)-\frac{b}{2(1+a^2)}\left\{\ln\left[b^2-2abz+(1+a^2)z^2\right]+2a\tan^{-1}\left[\left(\frac{1+a^2}{b}\right)z-a\right]\right\}\right]_h^{h+t}$$

And upon substituting the values of a and b this is equal to

$$(h+t)\tan^{-1}\left(\frac{t.\cot\alpha-x'}{h+t}\right)+h\tan^{-1}\frac{x'}{h}$$
$$-\frac{\sin^2\alpha}{2}(h\cot\alpha+x')\left[\ln\left\{(x'-t\cot\alpha)^2+(h+t)^2\right\}-\ln(x'^2+h^2)\right.$$
$$-2\cot\alpha.\tan^{-1}\left\{\frac{x'\cot\alpha-h-t\,\mathrm{cosec}^2\alpha}{h\cot\alpha+x'}\right\} \qquad (7)$$
$$\left.+2\cot\alpha.\tan^{-1}\left\{\frac{x'\cot\alpha-h}{h\cot\alpha+x'}\right\}\right]$$

So referring back to (4) we get:

$$g(x')=2G\rho\left[\frac{\pi t}{2}+(h+t)\tan^{-1}\frac{x'-t\cot\alpha}{h+t}-h\tan^{-1}\frac{x'}{h}\right.$$
$$+\frac{\sin^2\alpha}{2}(h\cot\alpha+x')\left\{\ln\left[\frac{(x'-t\cot\alpha)^2+(h+t)^2}{x'^2+h^2}\right]\right.$$
$$\left.\left.-2\cot\alpha\left(\tan^{-1}\frac{x'\cot\alpha-h-t\,\mathrm{cosec}^2\alpha}{x'+h\cot\alpha}-\tan^{-1}\frac{x'\cot\alpha-h}{x'+h\cot\alpha}\right)\right\}\right] \qquad (8)$$

Alternative derivation of $\int_h^{h+t} \tan^{-1}\left(a - \frac{b}{z}\right) dz$

Let $u = a - \frac{b}{z}$ so $du = \frac{b.dz}{z^2}$ and $dz = \frac{b.du}{(a-u)^2}$, and integrating by parts we get:

$$\int_{a-\frac{b}{h}}^{a-\frac{b}{h+t}} \frac{b}{(a-u)^2} \tan^{-1} u . du = b\left[\frac{1}{a-u} \tan^{-1} u\right]_{a-\frac{b}{h}}^{a-\frac{b}{h+t}} - b\int_{a-\frac{b}{h}}^{a-\frac{b}{h+t}} \frac{du}{(a-u)(1+u^2)} \quad (9)$$

The first term $= (h+t)\tan^{-1}\left(a - \frac{b}{h+t}\right) - h\tan^{-1}\left(a - \frac{b}{h}\right)$

which upon substituting for a and b becomes

$$-(h+t)\tan^{-1}\frac{x' - t\cot\alpha}{h+t} + h\tan^{-1}\frac{x'}{h} \quad (10)$$

The second term becomes (letting $u_1 = a - \frac{b}{h}$ and $u_2 = a - \frac{b}{h+t}$)

$$-\frac{b}{1+a^2}\int_{u_1}^{u_2}\left(\frac{1}{a-u} + \frac{u}{1+u^2} + \frac{a}{1+u^2}\right) du$$

$$= -\frac{b}{1+a^2}\left[-\ln(a-u) + \frac{1}{2}\ln(1+u^2) + a\tan^{-1} u\right]_{u_1}^{u_2}$$

$$= -\frac{b}{1+a^2}\left[\ln\left(\frac{a-u_1}{a-u_2}\right) + \frac{1}{2}\ln\left(\frac{1+u_2^2}{1+u_1^2}\right) + a(\tan^{-1}u_2 - \tan^{-1}u_1)\right]$$

And substituting for u_1, u_2, a and b this equals

$$-\frac{\sin^2\alpha}{2}(h\cot\alpha + x')\left[\ln\frac{(x' - t\cot\alpha)^2 + (h+t)^2}{x'^2 + h^2} - 2\cot\alpha\left(\tan^{-1}\frac{x' - t\cot\alpha}{h+t} - \tan^{-1}\frac{x'}{h}\right)\right] \quad (11)$$

Thus substituting (10) and (11) in (9) and then substituting this in (4);

$$g(x') = 2G\rho\left[\frac{\pi t}{2} + (h+t)\tan^{-1}\frac{x'-t\cot\alpha}{h+t} - h\tan^{-1}\frac{x'}{h}\right.$$

$$\left. + \frac{\sin^2\alpha}{2}(h\cot\alpha + x')\left[\ln\frac{(x'-t\cot\alpha)^2+(h+t)^2}{x'^2+h^2} - 2\cot\alpha\left(\tan^{-1}\frac{x'-t\cot\alpha}{h+t} - \tan^{-1}\frac{x'}{h}\right)\right]\right] \quad (12)$$

Comparing equations (8) and (12) it will be noted that they are not identical. Both equations do however yield exactly the same results. Use of equation (12) is recommended as it is slightly simpler. Both equations have been derived for the sake of verification only. Equation (12) compares favourably with equation 10-4 quoted by Grant and West, without putting $t = 1$.

Note also that when $\alpha = 90$ degrees, these equations reduce to the equation given in appendix II.